Model Questions and Answers on Marine Hydrodynamics

By

T. SAHOO

Dept. of Ocean Engineering and Naval Architecture, IIT, Kharagpur, INDIA

tsahoo1967@gmail.com

Kinematics of fluid flows

1. The velocity field associated with a fluid flow given by $u = 20y^2,\ v = -20xy,\ w = 0.$ Find the acceleration, the angular velocity and the vorticity vector at the point (1,-1,2).

   The acceleration vector is given by

   $$\vec{a} = \frac{D\vec{q}}{Dt} = \frac{\partial \vec{q}}{\partial t} + u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + w\frac{\partial \vec{q}}{\partial z}$$

   $$\text{Thus, } \vec{a} = 20y^2(-20y\hat{j}) - 20xy(40y\hat{i} - 20x\hat{j})$$
   $$= -800xy^2\hat{i} - 400(y^3 - x^2y)\hat{j}.$$

   Thus at (1,-1, 2), $\vec{a}$ is given by $\vec{a} = 800\hat{i}m/s^2$

   The vorticityvector is given by

   $$\vec{\Omega} = (\Omega_x,\ \Omega_y,\ \Omega_z)$$
   $$= \left(\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right),\ \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right)$$
   $$= \big(0,\ 0,\ (-20y - 40y)\big) = (0\ ,0,\ -60y)$$

   Hence, at $(1, \text{-}1, 2), \Omega_z = (0,\ 0,\ 60).$

   Thus, the x and y component of vorticity vector vanish whilst the z-component of vorticity vector is given by $\vec{\Omega} = \Omega_z\hat{k} = 60\hat{k}\,rad/s.$ Thus, the angular velocity at $(1, -1,\ 2)$ is given by $(0,\ 0,\ 30).$

Application of continuity equation

2. Water flows at a uniform speed of $5m/s$ into a nozzle whose diameter reduces from $10cm$ to $2cm$. Find the flow velocity leaving the nozzle and the flow rate.

   Ans: From continuity equation, we have

   Given $q_1 = 5m/s, d_1 = 01m, d_2 = 0.02m$

   $$\text{Thus, } A_1 = \pi\left(\frac{0.1}{2}\right)^2 m^2,\ A_2 = \pi\left(\frac{0.02}{2}\right)^2 = \pi(0.01)^2 m^2.$$

   Hence, from continuity equation $A_1q_1 = A_2q_2$ it is derived that

$\pi(0.01)^2 \times q_2 = \pi(0.05)^2 \times 5$

$\Rightarrow q_2 = 125 m/s.$

Further, flow rate near the nozzle is obtained as

$Q = A_1 q_1 = 0.0125\pi m^3/s.$

Example of continuity equation in Cartesian co-ordinate system

3. Show whether the velocity field associated with

$u = \frac{ay}{x^2+y^2},\ v = -\frac{ax}{x^2+y^2},\ w = 0.$ Represent the flow of an incompressible fluid.

Ans: For incompressible fluid flow $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} = 0,$ where $\frac{\partial w}{\partial x} = 0$

Substituting for u, v and $w,$ we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(\frac{ay}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{ax}{x^2+y^2}\right) = -\frac{2axy}{\left(x^2+y^2\right)^2} + \frac{2axy}{\left(x^2+y^2\right)^2} = 0.$$

Hence, the given velocity field is a possible incompressible flow.

4. The x-component of the velocity field of an incompressible flow is given by $u = Ay$. Determine $v(x, y)$ if $u(x, y) = 0.$

Ans: The continuity equation, in case of two-dimensional flow is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = 0 \quad (\text{as } u = Ay)$$

$\Rightarrow$ v=k(x) (an arbitrary function of x)

$\Rightarrow$ v(x,0)=0=k(x) $\Rightarrow$ v(x,y)=0

NB: It is clear that for non-zero v, u(x,y) would have to vary with x or v(x,0) would have to be non-zero. This is the case of a flow which is rotational as the z-component of vorticity vector is $uy - vx = A \neq 0$.

5. Given the x- component of the velocity of an incompressible plane flow by

$u = 5 + \frac{ax}{x^2+y^2}$. Determine $v(x, y)$ assuming $v(x, 0) = 0$.

Ans: $\frac{\partial u}{\partial x} = \frac{a\left(x^2+y^2\right) - 2ax^2}{\left(x^2+y^2\right)^2} = \frac{a\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$

Therefore, for an incompressible fluid, $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$

Now $\frac{\partial v}{\partial y} = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} \Rightarrow v = a\int \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} dy + c \; = a\frac{y}{x^2 + y^2} + c$

Thus, $v(x,0) = 0 \Rightarrow c = 0 \Rightarrow v = \frac{ay}{x^2 + y^2}$.

Example of continuity equation in cylindrical polar co-ordinate system

6. Show that the vector field

$q_r = a\left(1 - \frac{1}{r^2}\right)\cos\theta,\; q_\theta = -a\left(1 + \frac{1}{r^2}\right)\cos\theta\sin\theta,\; q_z = 0$ represent a possible flow.

Ans. From equation of continuity, $\frac{1}{r}\frac{\partial}{\partial r}(q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(q_\theta) + \frac{\partial}{\partial z}(q_z) = 0$.

Here, $\frac{1}{r}\frac{\partial q_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z}$

$$= \frac{a}{r}\frac{\partial}{\partial r}\left\{r\left(1 - \frac{1}{r^2}\right)\right\}\cos\theta - \frac{a}{r}\left(1 + \frac{1}{r^2}\right)\cos\theta$$

$$= \frac{a}{r}\left(1 + \frac{1}{r^2}\right)\cos\theta - \frac{a}{r}\left(1 + \frac{1}{r^2}\right)\cos\theta = 0.$$

Hence, the vector field represents a possible flow.

Axi-symmetric flow and continuity equation in spherical polar coordinate

7. Show that the velocity field given by $q_r = a\left(1 - \frac{8}{r^3}\right)\cos\theta,\; q_\theta = -a\left(1 + \frac{4}{r^3}\right)\sin\theta,\; q_\varphi = 0$

represents the flow of an incompressible fluid.

Ans: The continuity equation for any symmetric flow in spherical and polar coordinate is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(q_\theta \sin\theta) = 0.$$

Substituting for q_r, q_θ and q_φ, it is clear that the continuity equation is satisfied.

Axisymmetric irrotational flow in cylindrical co-ordinate system

In cylindrical polar coordinate system, $x = r\cos\theta, y = r\sin\theta, z = z,$ where r is the distance from origin and r is the radial distance from z-axis. The bodies of revolution coincide with z-axis and $u_\theta = 0$. Further, $u_r = \frac{\partial \phi}{\partial r},\; u_z = \frac{\partial \phi}{\partial z}$.

Thus, in cylindrical co-ordinates system, the continuity equation reduces to

$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\partial u_z}{\partial z}=0$ and in case of irrotational motion, ϕ satisfies

$$\nabla^2\phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial z^2}=\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad \text{(A)}$$

The vorticity vector is given by $w_\varphi=\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}$.

Further, stream function and velocity potential for axisymmetric flow are related by

$$u_z=\frac{1}{r}\frac{\partial\psi}{\partial r},\quad u_r=-\frac{1}{r}\frac{\partial\psi}{\partial z},\quad u_\theta=0 \qquad \text{(B)}$$

where ψ is the stream function.For irrotational flow

$$w_\theta=\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}=0 \qquad \text{(C)}$$

which using (B) yields

$$\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=0$$

which is different from Laplace equation satisfied by ϕ in (A).

Axi-symmetric flow (streamlines are conical surface)

8. Assuming the spherical coordinate system, discuss the flow associated with the velocity vector is given by $u_r=\frac{Q}{4\pi r^2},\ \ u_\theta=0.$

Ans: Given velocity vector $u_r=\frac{Q}{4\pi r^2},\ \ u_\theta=0$

Thus, $u_r=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}=\frac{\partial\phi}{\partial r}=\frac{Q}{4\pi r^2}$ (A)

$u_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}=\frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial r}=0$ (B)

From (A), $\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}=\frac{Q}{4\pi r^2}\Rightarrow\frac{\partial\psi}{\partial\theta}=\frac{Q\sin\theta}{4\pi}\Rightarrow\psi=\frac{-Q\cos\theta}{4\pi}+f(r)$

Further, $\frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial r}=0\Rightarrow g(\theta)=\frac{-Q\cos\theta}{4\pi},\quad f(r)=0\Rightarrow\psi=g(\theta)$

Therefore, $\psi=\frac{-Q\cos\theta}{4\pi}$

Further, $\frac{1}{r}\frac{\partial\phi}{\partial\theta}=0\Rightarrow\phi=\bar{f}(r)$

Again, $\frac{\partial\phi}{\partial r}=\frac{Q}{4\pi r^2}\Rightarrow\phi=\frac{-Q}{4\pi r}+g(\theta)$

Therefore $\phi = \frac{-Q}{4\pi r}$.

Thus equipotential surfaces are spherical shells and streamlines are conical surfaces on which $\theta =$ constant.

Basic difference between a plane flow and an axi-symmetric flow.

Lines of constant ϕ and lines of constant ψ are not orthogonal for axi-symmetric flows for irrotational motion, whilst lines of constant ϕ and lines of constant ψ are orthogonal for plane flows. Thus, in plane polar coordinate in case of irrotational motion, the stream function satisfy the Laplace equation whilst in axi-symmetric coordinate system, stream function does not satisfy the Laplace equation. Table shows the basic differences among various types of flows.

	2-D Cartesian co-ordinate system	Polar Coordinate system	Axi-symmetric cylindrical polar Co-ordinate system
Co-ordinates	(x,y)	$x = r\cos\theta, y = r\sin\theta$	$x = r\cos\theta, y = r\sin\theta, z = z$
Velocity	(u,v)	$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, u_\theta = -\frac{\partial\psi}{\partial r}$	$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z},\ u_\theta = 0,\ u_z = \frac{1}{r}\frac{\partial\psi}{\partial r}$,
Irrotational motion	$u_x = v_y, u_y = -v_x$	$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r}$, $u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$	$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} = \frac{\partial\phi}{\partial r}$, $u_z = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{\partial\phi}{\partial z}$
Laplace equation	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} = 0$
Vorticity	$\boldsymbol{\Omega} = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)$	$\boldsymbol{\Omega} = -\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right\}$	$\boldsymbol{\Omega} = \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}$

Irrotational flow, velocity potential and stream functions

9. Show that the streamlines associated with the flow whose velocity potential is $\phi = A\tan^{-1}(x/y)$ are circular.

Ans.Given $\phi = A\tan^{-1}(x/y)$

Thus, the relation between velocity potential and stream function yields

$$\frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = -\frac{Ay}{x^2 + y^2}$$

which on integration gives $\psi = -\frac{A}{2}\ln(x^2 + y^2) + f(x)$.

Further, $\frac{\partial\phi}{\partial y}=\frac{\partial\psi}{\partial x}=-\frac{Ax}{x^2+y^2}+\frac{df}{dx}=-\frac{Ax}{x^2+y^2}$

which yields $\frac{df}{dx}=0.$

Thus, $f=\text{constant}$

$=0$, without loss of generality

Hence Therefore, $\psi(x,y)=-\frac{A}{2}\ln(x^2+y^2).$

Thus, the streamlines are given by $\psi(x,y)=\text{constant}$ which yields $x^2+y^2=\text{constant}.$

Thus, the streamlines are concentric circles with centre at origin.

10. The velocity potential for a two dimensional fluid flow is given by $\phi=(x-t)(y-t).$ Find the streamlines at time t for the flow.

Ans: We have $u=\phi_x=\psi_y=(y-t)\Rightarrow\psi=(y-t)^2/2+f(x)$

Further, $v=\phi_y=-\psi_x=f^{'}(x)$

Thus, $\phi_y=(x-t)\Rightarrow\psi_x=-(x-t),$

which yields $\psi=-(x-t)^2+g(y).$

Therefore, $\frac{(y-t)^2}{2}+f(x)=-\frac{(x-t)^2}{2}+g(y)$

$\Rightarrow(x-t)^2+(y-t)^2=\text{constant}$

Thus $(x-t)^2+(y-t)^2=\text{constant}$ yield the streamlines.

11. The velocity potential for a flow is given by $\phi(x,y,t)=(-3x+5y)\cos\omega t$ where ω is a constant. Determine the stream function for the flow.

Ans: Given $\phi=(-3x+5y)\cos\omega t$

Thus, $\psi_y=\phi_x=-3\cos\omega t$

$\Rightarrow\psi=-3y\cos\omega t+f(x)$

Further, $\psi_x=-\frac{\partial\phi}{\partial y}=5\cos\omega t=f^{'}(x)$

$\Rightarrow f(x)=5x\cos\omega t+K$ (K is arbitary and is chosen as zero)

Thus, the stream function is given by $\psi=(5x-3y)\cos\omega t.$

12. The stream function for a two-dimensional incompressible flow is $\psi=\frac{ax^2}{2}+bxy-\frac{cy^2}{2},$ where a, b and c are known constants. Find the condition for the flow to be irrigational and thus find the velocity potential for the flow.

Ans: Given the stream function $\psi = \frac{ax^2}{2} + bxy - \frac{cy^2}{2}$,

Now, $\frac{\partial^2\psi}{\partial x^2} = a, \quad \frac{\partial^2\psi}{\partial y^2} = -c.$

Therefore, for the flow to be irrigational,

$\nabla^2\psi = 0 \Rightarrow a - c = 0$

$\Rightarrow a = c.$

Thus, for irrotational motion, $\mathrm{a = c}$. Assuming the flow as irrotational, there exists a velocity potential $\phi(\mathrm{x,y})$ such that $\phi_x = \psi_y = bx - cy \Rightarrow \phi = \frac{bx^2}{2} - cxy + f(y).$

Further, $\phi_y = -cx + f'(y) = -\psi_x = -(ax + by) \Rightarrow \phi = -axy + \frac{by^2}{2} + g(x).$

Thus, $g(x) = \frac{bx^2}{2}, \ f(y) = \frac{by^2}{2}$.

Hence, $\phi(x, y) = \frac{b}{2}(x^2 + y^2) - axy$ is the required velocity potential.

13. Suppose the stream function are given by $\psi(x, y) = xy$ which represent flow around a rectangular corner. Find the velocity potentials for the flow if exist.

Ans: Given $\psi(x, y) = xy$

Therefore $\nabla^2\psi = 0$

Thus, the flow is irrotational.

Thus, there exists a velocity potential ϕ which will satisfy

$u = \phi_x = \psi_y = x$

$\Rightarrow \phi = x^2/2 + f(y)$ (A)

$v = \phi_y = -\psi_x = -y$

$\Rightarrow \phi = -y^2/2 + g(x)$ (B)

From (A) and (B),

$f(y) = -y^2/2, \quad g(x) = x^2/2$

Thus $\phi = \frac{1}{2}(x^2 - y^2)$

14. The velocity components associated with the two dimensional flow of an inviscid fluid are $u = \frac{kx}{x^2 + y^2}$, $v = \frac{ky}{x^2 + y^2}$. Is the flow irrotational? Find the streamline passing through the points (1,0) and (3,0).

Ans: The fluid is irrotational, if $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

Now $\frac{\partial u}{\partial y} = \frac{-2kxy}{(x^2 + y^2)^2} = \frac{\partial v}{\partial x}$. (A)

Thus, the flow is irrotational.

Next, assume that ψ is the stream function, thus

$$u = \frac{\partial \psi}{\partial y} = \frac{-kx}{x^2 + y^2}$$

$$\Rightarrow \psi = -k \tan^{-1}\left(\frac{y}{x}\right) + f(x) \qquad \text{(B)}$$

where $f(x)$ is an arbitrary function.

Further, from (A) and (B),

$$v = \frac{-\partial \Psi}{\partial x} = \frac{ky}{x^2 + y^2} = \frac{ky}{x^2 + y^2} + f'(x)$$

$$\Rightarrow f'(x) = 0 \Rightarrow f(x) = c \text{ (A constant)} \qquad \text{(C)}$$

Hence, (B) and (C) yields

$$\psi = -k \tan^{-1}\left(\frac{y}{x}\right) + c$$

For stream lines, $\psi =$ constant

$$\Rightarrow \tan^{-1}\frac{y}{x} = K \text{ (A constant)} \Rightarrow y = x \tan K = kx, \quad \text{(K being a constant)} \qquad \text{(D)}$$

Eq.(D) is the required streamline. Since the streamline passes through (1, 0), Eq.(D) yield $k = 0$. Thus, $y = 0$ is the streamline passing through (1, 0). Further, since the streamline passes through (3, 2), Eq.(D) yields $y = (2/3)x$ as the other streamline.

15. The stream function associated with a flow field is given by $\psi(x, y) = xy$. Prove that the motion is irrotational. Find the components of velocity and hence find the velocity potentials.

Ans: Given $\psi(x, y) = xy \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

Thus, the flow is irrotational. Let u and v bethe x and y components of velocity. Thus $u = \psi_y = x$ and $v = -\psi_x = -y$.

Thus $u = \phi_x = x \Rightarrow \phi = x^2/2 + g(y)$ (A)

$v = \phi_y = -y \Rightarrow \phi = -y^2/2 + f(x).$ (B)

From (A) and (B), the velocity potential ϕ is obtained as $\phi(x, y) = (x^2 + y^2)/2$.

16. Describe necessary condition for the $\phi = ax^2 + by^2 + cz^2$ to be the velocity potential of anirrotational motion of a fluid flow.

Ans: Let ϕ to be the velocity potential, of an irrotational motion of a fluid, then

$\nabla^2\phi = 0$ (A)

Given $\phi = ax^2 + by^2 + cz^2$ (B)

Equation (A) and (B) yields, $a + b + c = 0$ which is the necessary condition.

Relation between velocity potential and stream function in polar co-ordinate

17. Find the stream function associated with the two-dimensional incompressible flow with velocity components given by $v_r = u\left(1 - \frac{a^2}{r^2}\right)\cos\theta,\ v_\theta = -u\left(1 + \frac{a^2}{r^2}\right)\sin\theta$. Hence, obtain the stream lines.

Ans: Let ψ be the stream function associated with the flow.

Since, $u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \Rightarrow \frac{\partial\psi}{\partial\theta} = ru_r$

$\Rightarrow \psi = \int ru_r d\theta = u\left(r - \frac{a^2}{r}\right)\sin\theta + f(r)$

Further, $\frac{\partial\psi}{\partial r} = u\left(1 + \frac{a^2}{r^2}\right)\sin\theta + f^{'}(r)$

Now $v_\theta = -\frac{\partial\psi}{\partial r} = -u\left(1 + \frac{a^2}{r^2}\right)\sin\theta + f^{'}(r) = -u\left(1 + \frac{a^2}{r^2}\right)\sin\theta$

$\Rightarrow f^{'}(r) = 0, \ \Rightarrow f(r) = \text{constant} = 0$ (w.l.g.)

$\Rightarrow \psi = u\left(r - \frac{a^2}{r}\right)\sin\theta$

Thus, $u\left(r - \frac{a^2}{r}\right)\sin\theta = \text{constant}$ gives the streamline for the flow.

Application of Bernoulli's equation in steady state and vortex motion

18. Assuming that the pressure far from a tornado in the atmosphere is zero gauge. If the velocity at $\text{r} = 20\text{m}$. in 20m/s find the velocity and pressure at $\text{r} = 2\text{m}$ (Hint: Assume that the tornado is modeled as anirrotational vortex with density of air $\rho = 1.2\text{kg/m}^3$).

Ans.In case of the tornado, it is assumed that the flow is circular in nature with $v_r = 0, v_z = 0.$

Further, it is given that $r = 20m, v_\theta = 20m/s.$

Thus, $v_\theta = -\Gamma/2\pi r$

$$\Rightarrow \Gamma = 2\pi r v_\theta = 2\pi \times 20 \times 20 = -800\pi m^2/s$$

Hence, the velocity at $r = 2m$ is given by

$$v_\theta = \frac{\Gamma}{2\pi r} = \frac{-800\pi}{2\pi \times 2} = 200m/s.$$

Assuming thatthe motion is irrotational, the pressure is given by

$$p = -\frac{v_\theta^2}{2} = 24000p_a.$$

The negative sign in pressure refers to vacuum. This negative pressure which creates a vacuum causes the roots of building to blow off during a tornado.

Two dimensional irrotationalflow and streamlines

19. Discuss whether the flow is irrotational.

$$\vec{q} = \left(\frac{-ay}{x^2 + y^2}, \frac{-ax}{x^2 + y^2}, 0 \right)$$

Ans. It is easily verified that

$\frac{\partial}{\partial y}\left(\frac{-ay}{x^2 + y^2} \right) - \frac{\partial}{\partial x}\left(\frac{-ax}{x^2 + y^2} \right) = 0$ except at origin. Thus flow is irrotational except at $(x, y) = (0, 0).$

20.The velocity field associated with an irrotational incompressible fluid flow in 2-D given by $u = 2x,\ v = -ty$ where x and y are in meters and t is in seconds. Find the equation of stream line passing through (2,-1) at $t = 4s$.

Ans: Let ϕ be the velocity potential and ψ be the stream function associated with the flow field. Thus, the equation of streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dx}{x} = \frac{dy}{-2y}$$

Integrating both sides, it is derived that

$$\ln x + \frac{1}{2}\ln y = \text{constant}$$

$$\Rightarrow \ln x^2 y = \text{constant}$$

$$\Rightarrow x^2 y = c \text{ (a constant)}.$$

Since, the streamline passes through (2, -1), it is derived that $c = -4.$

Hence, the streamline passing through (2,-1) has the equation $x^2 y = -4.$

21. Determine the condition for which the velocity vector $u = ax + by,\ \ v = cx + dy$ will represent the flow of an incompressible fluid. Show that the streamlines of this motion are conic sections in general and rectangular hyperbolas when the motion is irrotational.

Ans: Given $u = ax + by,\ \ v = cx + dy$

$\Rightarrow \dfrac{\partial u}{\partial x} = a$ and $\dfrac{\partial v}{\partial y} = d$

For possible fluid motion, $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$, which yield $a + d = 0$

Now for the flow to be irrotational, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Therefore, the velocity field will represent an irrotational motion of a fluid for $a = d$ and $b = c.$ Next, to find the stream function ψ, we have

$$u = \frac{\partial \psi}{\partial y} = ax + by \Rightarrow \psi = axy + \frac{by^2}{2} + f(x)$$

Further $v = \dfrac{-\partial \psi}{\partial x} = cx + dy \Rightarrow \psi = -\left(\dfrac{cx^2}{2} + dxy\right) + g(y)$

Comparing the two expressions for the stream function ψ, it is derived that

$\psi = \dfrac{by^2}{2} - axy + \dfrac{bx^2}{2}$ which represents a rectangular hyperbola.

22. Show that the velocity field given by $u = 2cxy,\ \ v = c(a^2 + x^2 - y^2)$ represent the velocity vector of an incompressible fluid flow. Hence, determine the stream functions and discuss

Ans: Given $u = 2cxy,\ \ v = c(a^2 + x^2 - y^2)$

Thus, $\dfrac{\partial u}{\partial x} = 2cy$, $\dfrac{\partial v}{\partial y} = -2cy$

Now, $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

which ensures u and v are the velocity vector of an incompressible fluid flow.

Now $u = \dfrac{\partial \psi}{\partial y} = 2cxy \Rightarrow \psi = cxy^2 + f(x)$

Further, $v = -\dfrac{\partial \psi}{\partial x} = c(a^2 + x^2 - y^2)$

$$\Rightarrow \psi = c(a^2 x + \frac{x^3}{3} - y^2 x) + f(y)$$

Hence $f(x) = -a^2cx - \frac{cx^3}{3}$ and $g(y) = 0$

Therefore, $\psi = -cx(a^2 - \frac{x^2}{3} - y^2)$

Thus, $x(a^2 - \frac{x3}{3} - y^2) =$ constant yield the streamlines for the flow.

Two dimensional flow, stream function and vorticity vector

23. Find the relation between stream function and vorticity vector.

Ans. The vorticity vector $\vec{\Omega}$ in a two-dimensional flow has the component

$$\vec{\Omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = \left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right)\hat{k} = -(\nabla^2 \psi)\hat{k}$$

NOTE: When the flow is irrotational, then only $\vec{q} = \text{grad}\ \phi$, so that $u = \phi_x$ and $v = \phi_y$ leading to $\nabla^2 \psi = 0$ and $\nabla^2 \phi = 0$.

24. Show that the vorticity vector for any fluid flow satisfies satisfies $\nabla \cdot \vec{\Omega} = 0$

Ans: $\vec{\Omega} = (\Omega_{x,}\ \Omega_{y,}\ \Omega_z) = \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y},\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$

Therefore $\nabla \cdot \vec{\Omega} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$

Streamlines and flow with constant vorticity

25. Find the streamlines for the flow with constant vorticity.

Ans: The vortcity vector in the two-dimensional flow is given by

$$\Omega = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\hat{k} = \left\{\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{-\partial \psi}{\partial x}\right)\right\}\hat{k} = \left(\nabla^2 \psi\right)\hat{k}$$

Further, for $z = x + iy,\ \overline{z} = x - iy,$

$\psi(x, y)$ can be represented as $\psi(z, \overline{z})$.

Thus, $\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z} \cdot \frac{\partial z}{\partial x} + \frac{\partial \psi}{\partial \overline{z}} \cdot \frac{\partial \overline{z}}{\partial x} = \frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial \overline{z}}$

and $\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial z} \cdot \frac{\partial z}{\partial y} + \frac{\partial \psi}{\partial \overline{z}} \cdot \frac{\partial \overline{z}}{\partial y} = i\left(\frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial \overline{z}}\right)$

Thus, $2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y},\quad 2\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$

Hence, $\dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial x^2}=\left(\dfrac{\partial}{\partial x}-i\dfrac{\partial}{\partial y}\right)\left(\dfrac{\partial}{\partial x}+i\dfrac{\partial}{\partial y}\right)\psi=4\dfrac{\partial^2\psi}{\partial z\partial\overline{z}}$

Thus, flow with constant vorticity yields

$\Omega=\text{constant}$

$=\nabla^2\psi=4\dfrac{\partial^2\psi}{\partial z\partial\overline{z}}$

$\Rightarrow\psi=f(z)+\overline{f}(\overline{z})+\dfrac{1}{4}\Omega z\overline{z}$

where f is an arbitrary function of z.

Hence ψ can be rewritten as

$\psi=\dfrac{\Omega}{4}z\overline{z}+2if(z)$

Thus, the flow with constant vorticity ψ consists of flow whose stream function is $\dfrac{\Omega}{4}z\overline{z}$ along with an irrotational motion whose complex potential is $2if(z)$.

26. The velocity component of a two-dimensional inviscid incompressible flow are given by $u=2y+\dfrac{y}{\left(x^2+y^2\right)^{1/2}}$, $v=-2x-\dfrac{x}{\left(x^2+y^2\right)^{1/2}}$. Find the stream function and vorticity vector.

Ans: Let ψ be the stream function.

Then, $u=\psi_y=2y+\dfrac{y}{\left(x^2+y^2\right)^{1/2}}$,

which on integration yield $\psi=y^2+\displaystyle\int\dfrac{y}{\left(x^2+y^2\right)^{1/2}}+f(x)=y^2+(x^2+y^2)^{1/2}+f(x)$.

Further, $v=-\psi_x=-2x-\dfrac{x}{\left(x^2+y^2\right)^{1/2}}$, $\Rightarrow$ $\psi=x^2+\left(x^2+y^2\right)^{1/2}+g(y)$.

From the above two expressions for $\psi(x,y)$, we have, $f(x)=x^2$, $g(y)=y^2$

Hence $\psi=x^2+y^2+\left(x^2+y^2\right)^{1/2}$.

Thus, $x^2+y^2+\left(x^2+y^2\right)^{1/2}=K$ (a constant) gives the streamlines.

Next, the vorticity vector is given by

$\vec{\Omega}=\Omega_z\hat{k}=\left(\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial x}\right)\hat{k}=-\left\{4+\dfrac{1}{\left(x^2+y^2\right)^{1/2}}\right\}\hat{k}.$

27. Find the vorticity associated with the velocity field $\vec{q} = (-ay, ax, 0)$ and thus show that the vorticity vector associated with the flow constant.

Ans: Equations of streamlines are $\dfrac{dx}{-ay} = \dfrac{dy}{ax} = \dfrac{dz}{0}$

$\Rightarrow x^2 + y^2 = \text{constant}$ are the streamlines.

Further, $\vec{\Omega} = \text{curl}(\vec{q}) = \left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right)\hat{k} = 2a\hat{k}$

Thus, the vorticity vector for the flow remains the same everywhere. Here, the flow is rotational.

Streamlines are same does not imply motions are same (rotational / irrotational)

28. Consider the flow field with velocity components being given by $u = -wy,\ v = wx$ and $w = 0$. Find the stream function and thus the streamlines. Draw the basic difference between this flow and the flow whose velocity potential is given by $\phi = m\ln r,\ \ r = \sqrt{x^2 + y^2}$.

Ans: For the first flow,

$u = wy,\ v = wx,\ w = 0$

Here, $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

Thus an incompressible fluid flow is possible.

However, $\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x} = 2w \neq 0$

Hence, the flow is not irrotational. Thus, velocity potential does not exist for the flow. Here, Eqn. of the streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or $$\frac{dx}{wy} = \frac{dy}{-wx} = \frac{dz}{0}$$

Ist two equation gives $x^2 + y^2 = \text{constant},\ z = \text{constant}.$

The particles are following a circular path with centre at origin. On the other hand, for the second flow

$$W = \phi + i\psi = ik\ln z = ik\ln r - k\theta,\ z = re^{i\theta}$$

$\Rightarrow \phi = k\theta,\ \ \psi = k\ln r$

Now, $\psi = \text{constant} \Rightarrow x^2 + y^2 = \text{constant}$ which are the streamlines.

Here, $\dfrac{\partial \psi}{\partial x} = \dfrac{x}{x^2 + y^2} \Rightarrow \dfrac{\partial^2 \psi}{\partial x^2} = \dfrac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2},$

$$\frac{\partial \psi}{\partial y} = \frac{y}{x^2 + y^2} \Rightarrow \frac{\partial^2 \psi}{\partial y^2} = \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2},$$

Therefore, $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0.$

Hence, the flow is irrotational. Thus, in the second case, the flow is irrotational and particles follow a circular path whilst in the first case, the fluid motion is rotational.

Irrotational vortex or potential vortex - irrotational flow except at origin - Example of a flow for which streamlines are circular and flow need not have vorticity everywhere

29. Discuss the flow characteristics for the velocity vector given by $u_\theta=\frac{c}{r},\ u_r=0,\ u_z=0.$

Ans. Here, the vorticity at any point in the flow is given by

$$w_z=\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)-\frac{1}{r}\frac{\partial u_r}{\partial\theta}=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{c}{r})-0=\frac{c}{r}.$$

Thus, the flow is irrotational everywhere except at the origin.

Now, around a contour of radius r, the circulation is

$$\Gamma=\int_0^{2\pi}ru_\theta d\theta=2\pi c$$

which shows circulation is a constant and independent of the radius.

Further, using Stokes theorem

$$\Gamma=\int_A\Omega.dA$$

For a contour enclosing the origin, since $\Gamma=2\pi c\neq 0$

$\int_A\vec{\Omega}.dA\neq 0\Rightarrow\vec{\Omega}\neq 0$ somewhere within the area enclosed by the contour.

Since Γ is independent of r, the contour can be shrink without altering Γ. Thus the area can be shrinking so that $\vec{\Omega}$ must be infinite in order to make $\vec{\Omega}.dA$ to be finite and non-zero. Thus, the flow represented by $u_\theta=c/r$ is irrotational everywhere except at the origin where the vorticity is infinite. Such a flow is called an irrotational vortex or potential vortex. It may be noted that around a closed curve not containing the origin, the circulation is zero. Further, the equations of streamlines are given by

$$\frac{dr}{r}=\frac{rd\theta}{u_\theta}=\frac{dz}{u_z},\text{or }\frac{dr}{0}=\frac{r^2d\theta}{c}=\frac{dz}{0}$$

$$\Rightarrow\ dr=0,\text{or }r=\text{constant}$$

Therefore the streamlines are circles. This example illustrates that circular streamlines do not imply that flow should have vorticity everywhere.

Rankine vortex: In case of a Rankine vortex, the vortex is assumed to be uniform within a core of radius R and zero outside the core. For example; vortices like bathtub vortex or an atmospheric cyclone have a core that rotates almost like a sold body which is approximately irrotational at far field. Here, a rotational core exists as the tangential vector in an irrotational vortex has an infinite velocity jump at origin.

Example of flow for which stream function exists but velocity potential does not exist.

30. Give an example of a flow for which the stream function exist but velocity potential does not exist.

Ans. Consider the flow field represented by the velocity vector

$\vec{q} = (\omega y, \omega x, 0)$

Here, $\text{curl}\vec{q} = 2\omega\hat{k} \neq 0$

Thus the flow is rotational in nature and thus the flow is not of potential type.

Hence, ϕ will not exist. On the other hand, the equation of streamlines are given by

$$\frac{dx}{-\omega y} = \frac{dy}{\omega x} = \frac{dz}{0}$$

which yields that the streamlines are given by

$x^2 + y^2 = \text{constant, z} = \text{constant.}$

Thus $\psi = x^2 + y^2$ is the required stream function

Flow field and vortex lines

31. The velocity vector in the flow field is given by $\vec{q} = \hat{i}(Az - By) + \hat{j}(Bx - Cz) + \vec{k}(Cy - Ax)$ with A, B and C being non-zero constants. Determine the equation of the vortex lines.

Ans: The velocity vector is given by $\vec{q} = \hat{i}(Az - By) + \vec{j}(Bx - Cz) + \vec{k}(Cy - Ax)$.

Thus, the vorticity vector is given by

$$\vec{\Omega} = \text{curl}\vec{q} = \nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Az - By & Bx - Cz & Cy - Ax \end{vmatrix} = 2C\ \hat{i} + 2A\,\hat{j} + 2B\ \hat{k}.$$

Therefore, $\Omega_x = 2C,\ \Omega_y = A,\ \Omega_z = B$

Thus, equations of vortex lines are $\frac{dx}{\Omega_x} = \frac{dx}{\Omega_y} = \frac{dx}{\Omega_z}$ (A)

Substituting for Ω_x, Ω_y and Ω_z, Eq.(A) yield

$x = \frac{C}{B}z + k,\ y = \frac{A}{B}z + k$ as the two infinite systems of vortex lines.

32. The velocity vector for an in compressible flow is given by $\vec{q} = (Az - By)\hat{i} + (Bx - Cz)\hat{j} + (Cy - Ax)\hat{k}$, where A, B and C are non-zero constants. Find theequation of vortex lines.

Ans: The velocity vector for an incompressible flow field is given by

$\vec{q} = (Az - By)\hat{i} + (Bx - Cz)\hat{j} + (Cy - Ax)\hat{k} \Rightarrow u = Az - By,\ v = Bx - Cz,\ w = Cy - Ax$

Hence, the components of vorticity vectors are

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 2C\,,\ \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 2A\,,\ \Omega_z = \frac{\partial v}{\partial z} - \frac{\partial u}{\partial y} = 2B$$

Therefore, the equation of vortex lines are given by

$$\frac{dx}{\Omega_y} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$

which yield

$$\frac{dx}{2C} = \frac{dy}{2A} = \frac{dz}{2B} \qquad \text{(A)}$$

Now $\frac{dx}{2C} = \frac{dy}{2A} \Rightarrow Ax - Cy = c_1$ (B)

Further $\frac{dy}{2A} = \frac{dz}{2B} \Rightarrow By - Az = c_2$ (C)

Thus, the vortex lines are the intersections of the straight lines given by Eq. (B) and Eq. (C).

Application of Bernoulli's equation in steady state

33 Assuming that the wind speed in a storm is given by $30\text{m}/\text{s}$, determine the force acting on the $2\text{m}\times 3\text{m}$ door facing the storm. The door is in a highrise building and the wind speed is not reduced due to ground effects. Use density of air $\rho = 1.2\text{kg}/\text{m}^3$ and the fluid is inviscid and incompressible.

Ans: Assuming the flow is steady and fluid is incompressible, Bernoulli'sequation yields

$$\frac{q_1^2}{2g} + \frac{p_1^2}{\rho g} + h_1 = \frac{q_2^2}{2g} + \frac{p_2^2}{\rho g} + h_2$$

It is assumed that $q_2 = 0$ (i.e., wind speed near the dooris $q_2 = 0$).

Further, it is assumed that $h_1 = h_2$ and $q_2 = 0$ and the atmospheric pressure $p_1 = 0$ which is constant otherwise. Thus,

$$\frac{p_2}{\rho g} = \frac{q_2^2}{2g} \Rightarrow p_2 = \frac{\rho q_2^2}{2} = 12\times 45\text{N/m}^2 = 540\text{N/m}^2.$$

Now, total force on the door $= p\times A = 540\times 2\times 3 = 3240\text{N}$.

Stream function and the complex potential

34. Find the complex velocity potential where the stream function ψ is defined by

$\psi(x,y) = 2x(x^2 - 3y^2) + (x^2 - y^2)/2 + \alpha xy.$

Ans: Given $\psi(x,y) = 2x(x^2 - 3y^2) + \frac{1}{2}(x^2 - y^2) + \alpha xy$

Thus, $u = \phi_x = \psi_y = -12xy - y + \alpha x$

$\Rightarrow \phi = -6x^2 y - xy + \alpha x^2 + f(y)$ (A)

Further, $v = \phi_y = \psi_x = -6x^2 - 6y^2 + x + \alpha y$

$\Rightarrow \phi = -6x^2 y - 3y^3 + xy + \alpha y^2 + g(x)$ (B)

From (A) and (B) $f(y) = 2y^3 - \alpha y^2$ and $g(x) = \alpha x^2$

which yield $\phi(x, y) = \alpha(x^2 - y^2) - 2y(3x^2 - y^2) - xy$

Therefore,

$$W(z) = \phi + i\psi = \alpha(x^2 - y^2) - 2y(3x^2 - y^2) - xy + i\{2x(x^2 - 3y^2) + \frac{1}{2}(x^2 - y^2) - \alpha xy\}$$

$$= (\alpha - i/2)(x^2 - y^2) - (1 - i\alpha)xy - 2y(3x^2 - y^2) + 2ix(x^2 - 3y^2)$$

$$= (\alpha - i/2)z\overline{z} - (1 - i\alpha)xy - 2(y^3 + x^3) - 6xy(x + iy)$$

This gives the required complex potential.

Complex potential (Doublets or flow around circles touching the x-axis)

35. Show that the complex potential $w(z) = ua^2/z$ is associated with flow whose streamlines are circles which touch the x-axis at origin. This is also referred as doublet.

Ans: Given the complex potential

$$W(z) = \frac{ua^2}{z} = \frac{ua^2 e^{-i\theta}}{r} \Rightarrow \psi = -\frac{ua^2 \sin\theta}{r} = -\frac{ua^2 y}{x^2 + y^2}$$

The streamlines $\psi =$ constant are circles which touch the x-axis at the origin. The motion is due to a doublet at origin.

Complex potential (streamlines are coaxial circles with center on y-axis and equipotentials are circles with center on x-axis which are orthogonal co-axial circles)

36a. Describe the flow for the complex potential given by $W(z) = \frac{k}{\pi}\tan^{-1}\frac{z}{c}$

Ans: Now $W(z) = \frac{k}{\pi}\tan^{-1}\frac{z}{c} \Rightarrow \phi + i\psi = \frac{k}{\pi}\tan^{-1}\frac{x + iy}{c}$

which gives $\frac{x + iy}{c} = \tan\frac{\pi}{k}(\phi + i\psi)$

Eliminating ϕ and ψ, it is derived that

$$x^2 + \left(y - c\coth\frac{2\pi\psi}{k}\right)^2 = c^2 \text{cosech}^2\frac{2\pi\psi}{k} \text{ and } \left(x + c\coth\frac{2\pi\phi}{k}\right)^2 + y^2 = c^2\text{cosech}^2\frac{2\pi\phi}{k}.$$

Thus, the curves $\phi =$ constant and $\psi =$ constant gives orthogonal co-axial circles, where the circles with center on y-axis are streamlines and circles with center on x-axis yield the equipotential surface.

Complex potential (Flow in a circular cylinder in the presence of source and sink)

36b. A source and a sink of strength m are placed at $(\pm a/2, 0)$ with a fixed circular boundary $x^2 + y^2 = a^2$. Find the streamlines for the flow.

Ans: From circle theorem we know if $f(z)$ is the complex velocity potential for the flow having no rigid boundaries and such that there is no flow singularities outside the circle $|z| = a$. Then an introducing a rigid circular cylindrical surface of section $|z| = a$ into the flow, the new complex velocity potential for the flow within the boundary becomes

$w(z) = f(z) + \bar{f}(a^2/z)$ for $|z| \le a$.

The complex potential for a source and a sink of strength m located at $(\pm a/2, 0)$ with no circular boundary is $f(z) = -m\ln(z - a/2) + m\ln(z + a/2)$. When a circular cylinder of radius $|z| = a$ is introduced into the flow with the boundary, the new complex potential becomes

$$w(z) = -m\ln(z - a/2) + m\ln(z + a/2) - m\ln\left\{\frac{a^2}{z} - \frac{a}{2}\right\} + m\ln\left\{\frac{a^2}{z} + \frac{a}{2}\right\}$$

$$= m\left\{\ln\left(x + \frac{1}{2}a + iy\right) - \ln\left(x - \frac{1}{2}a + iy\right) + \ln\left(2a + x + iy\right) - \ln\left(2a - x - iy\right)\right\}.$$

$$\text{Hence, } \psi = m\left\{\tan^{-1}\left(\frac{y}{x + \frac{1}{2}a}\right) - \tan^{-1}\left(\frac{y}{x - \frac{1}{2}a}\right) + \tan^{-1}\left(\frac{y}{2a + x}\right) + \tan^{-1}\left(\frac{y}{2a - x}\right)\right\}$$

$$= m\left\{\tan^{-1}\frac{4ay}{4a^2 - r^2} - \tan^{-1}\frac{ay}{r^2 - \frac{1}{2}a^2}\right\} = m\tan^{-1}\left\{\frac{5ay(r^2 - a^2)}{(4a^2 - r^2)(r^2 - \frac{1}{2}a^2) + 4a^2y^2}\right\}.$$

Now, $\psi = \text{constant}$ yield $(4a^2 - r^2)(r^2 - \frac{1}{2}a^2) + 4a^2y^2 = Ky(r^2 - a^2)$,

which are the streamline equations with K being an arbitrary constant.

Complex potential (flow around a rectangular corner)

37. Discuss the flow pattern for the complex potential $w(z) = z^2$.

Ans: Given $w(z) = z^2 \Rightarrow \phi + i\psi = x^2 - y^2 + 2ixy$

$\Rightarrow \psi = 2xy$

which yield $xy = \text{constant}$ as the streamlines.

Thus, streamlines are rectangular hyperbolas representing flow around a rectangular corner.

Complex potential and two dimensional flow (flow past a wedge)

38. Find the stream function and speed q for the complex velocity potential $w(z) = -ce^{-in\pi} z^{n+1},\ n = \alpha/\pi - \alpha$. Find c if $|\mathrm{q}| = \mathrm{U}$ for $\mathrm{r} = 1$.

Ans: Given $w(z) = -ce^{-in\pi} z^{n+1},\ n = \alpha/(\pi - \alpha)$

Thus, $w(z) = \phi + i\psi = -ce^{-in\pi} z^{n+1}$

$$= -ce^{in\pi}(re^{i\theta})^{n+1} = -ce^{-in\pi} r^{n+1} e^{i(n+1)} = -cr^{n+1} e^{i\{n(\theta-\pi)+\theta\}}.$$

Therefore, $\phi = -cr^{n+1}\cos\{n(\theta-\pi)+\theta\},\ \psi = -cr^{n+1}\sin\{n(\theta-\pi)+\theta\}.$

Hence $\psi = 0 \Rightarrow \theta = \alpha,\ 2\pi - \alpha$

Thus, the streamlines are the flow past a wedge of angle 2α whose section is placed symmetrically with z-axis. Further, the speed q is given by

$$q^2 = \left|\frac{dw}{dz}.\frac{d\overline{w}}{dz}\right| = c^2(n+1)^2 r^{2n}$$

Thus, for $r = 1,\ |q| = u$ then

$$u = c(n+1) \Rightarrow q = ur^n$$

Thus, the speed at a distance r from vortex is ur^n.

Complex potential (stagnation point flow)

39. Determine the flow near a stagnation point in the xy plane.

Ans: Let ϕ be the stream function and origin be a stagnation point.

Hence, at the origin

$$\frac{\partial\psi}{\partial x} = 0,\ \frac{\partial\psi}{\partial y} = 0$$

Let $\psi = 0$ be at the origin.

Thus, in the xy plane, when x, y are small, ψ can be expanded as

$$\psi = ax^2 + 2hxy + by^2 + cx + dy + e + \cdots$$

since ψ is assumed to be zero at origin, $e = 0$

Further, $\psi_x = 0,\ \psi_y = 0$ at origin yield $d = c = 0$

Therefore, $\psi = ax^2 + 2hxy + by^2$

Thus, for x, y, small, ψ is approximated as

$$\psi = ax^2 + 2hxy + by^2,$$

which represents two straight lines.

Further, when the flow is irrotational,

$$\nabla^2\psi = 0$$

which yields $a + b = 0.$

40. Find the stagnation point for the flow field given by $W(z) = 2z + 3iz^2$.

Ans: Given $W(z) = 2z + 3iz^2$

For stagnation point, $\frac{dW}{dz} = 0. \Rightarrow 2 + 6iz = 0 \Rightarrow z = \frac{i}{3}$.

Therefore, the point (0,1/3) is the stagnation point.

41. For the complex potential $w(z) = ua\left(z/a\right)^{\pi/\alpha}$, find the stagnation points.

Ans: For stagnation point $\frac{dw}{dz} = 0$.

Thus, for $w(z) = ua\left(z/a\right)^{\pi/\alpha}$, the stagnation points are for $z^{\frac{\pi}{\alpha}-1} = 0$

Therefore, if $\pi < \alpha$, the stagnation point is at infinity. On the other hand, if $\pi > \alpha$, the stagnation point is at $\mathrm{z} = 0$.

Superposition of a uniform flow and a source (half body)

42. Discuss the flow generated due to the superposition of a source of strength *m* and an uniformstream of speed *u*.

Ans: Here $W(z) = uz + m\ln z$

$\phi + i\psi = u(x + iy) + m(\ln r + i\theta)$

$\Rightarrow \psi = ur\sin\theta + m\theta,$ where $x = r\cos\theta, y = r\sin\theta$

Hence, $\frac{dW}{dz} = 0 \Rightarrow z = -\frac{m}{u}$

Therefore, the stagnation point occurs at $z = -\frac{m}{u}$.

Thus, the value of the stream function passing through the stagnation point is

$$\psi = uy + m\tan^{-1}\frac{y}{x}\bigg|_{z=-\frac{m}{u}} \Rightarrow \psi = ur\sin\theta + m\theta\Big|_{(r,\theta)=\left(\frac{m}{u},\pi\right)} = u\frac{m}{u}\sin\pi + m\pi = m\pi$$

Thus, the equation of the streamline passing through the point $z = -\frac{m}{u}$ is

$ur\sin\theta + m\theta = m\pi$ (A)

Eq.(A) represents a semi-infinite body with a smooth nose and is known as a half body.

Potential flow against a fixed plane wall

43.Show that the velocity potential $\phi = a(x^2 + y^2 - 2z^2)/2$ represents the flow against a fixed plane wall.

Ans: Given $\phi = \frac{a}{2}(x^2 + y^2 - 2z^2)$.

Thus, $u = \frac{\partial\phi}{\partial x} = ax,\ v = \frac{\partial\phi}{\partial y} = ay,\ w = \frac{\partial\phi}{\partial z} = -2az.$

Now, $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = a + a - 2a = 0$.

Thus the flow is irrotational.The equation of the streamlines associated with the flow satisfy $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$.

The equation $\frac{dy}{v} = \frac{dz}{w}$ yield

$\frac{dz}{-2z} = \frac{dy}{y} \Rightarrow y^2 z = k$ (a constant).

Further, the equation $\frac{dx}{u} = \frac{dy}{v}$ yield

$\frac{dx}{ax} = \frac{dy}{ay} \ \Rightarrow x = cy,$ with c being a constant.

The intersection of the two curves gives the streamlines. These streamlines are often called a cubic hyperbola and is the flow against a fixed plane wall.

Complex source potential and velocity potential for a flow

44. Show that the point source of strength on flow which is symmetrical in the radial direction is given by $W = \frac{m}{2\pi}\ln z$ has the velocity potential ϕ which satisfies

$\nabla^2\phi = m\delta(\overline{x})$, $\quad \vec{x} = (x, y)$

Ans: Given $W = \frac{m}{2\pi}\ln z = \frac{m}{2\pi}\ln re^{i\theta}$

$\Rightarrow \phi + i\psi = \frac{m}{2\pi}\ln r + \frac{im}{2\pi}\theta$

Therefore, $\phi = \frac{m}{2\pi}\ln r,\quad \psi = \frac{m\theta}{2\pi}$. Thus, the stream lines are given by $\theta = \mathrm{k(a\ constant)} \Rightarrow \tan^{-1}(y/x) = k \Rightarrow y = cx.$ Further, since $\oint \Delta\phi \cdot nds = \int_0^2 \frac{\partial\phi}{\partial r} rd\theta = m$, the source strength is the rate of production of fluid per unit span.

Further, ϕ satisfies $\nabla^2\phi = 0$ except at origin.

However, $\phi = \frac{m}{2\pi}\ln r = \lim_{\in\to 0}\frac{m}{4\pi}\ln(r^2 + \in^2)$

Thus, $\nabla^2\phi = \nabla^2\left(\frac{m}{2\pi}\ln r\right) = \lim_{\in\to 0}\frac{m}{4\pi}\nabla^2\{\ln(r^2+\in^2)\}$

$= \lim_{\in\to 0}\frac{m}{4\pi}\left\{\frac{\partial}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right\}\ln(r^2+\in^2) = \lim_{\in\to 0}\frac{\in^2}{\pi(r^2+\in^2)^2} = \delta(\overline{x})$

$= \frac{\pi}{\pi}\delta(x)\delta(y) = \delta(x)\delta(y) = \delta(\overline{x})$, where $\overline{x} = (x, y)$ and $r = \sqrt{(x-x_0)^2+(y-y_0)^2}$.

45. Show that the velocity potential $\phi(x, y) = \frac{m}{2\pi}\ln|x-y|$ produced due to a source of strength m satisfies $\nabla^2\phi = m\delta(x-y)$. The velocity potential ϕ is often known as the free space Green's function.

Ans: Proof follows from previous exercise.

Complex potential (flow in the presence of source and sink)

46. Consider a source and a sink each of strength m are located at distance c in either side of the origin.

Ans: Assuming that the source of strength m is located $(\mathrm{c}, 0)$ and sink of strength m is located at $(-\mathrm{c}, 0)$, the complex potential given by $\mathrm{w(z) = -m\ln(z+c) + m\ln(z-c)}$ which yields $\phi + i\psi = -\mathrm{m}\ln(\mathrm{x+c+iy}) + \mathrm{m}\ln(\mathrm{x-c+iy})$. Therefore, the stream function ψ is obtained as

$$\psi = \mathrm{m}\tan^{-1}\frac{\mathrm{y}}{\mathrm{x-c}} - \mathrm{m}\tan^{-1}\frac{\mathrm{y}}{\mathrm{x+c}} = \mathrm{m}\tan^{-1}\left(\frac{\frac{\mathrm{y}}{\mathrm{x-c}}-\frac{\mathrm{y}}{\mathrm{x+c}}}{1+\frac{\mathrm{y}^2}{\mathrm{x}^2-\mathrm{c}^2}}\right) = \mathrm{m}\tan^{-1}\frac{2\mathrm{cy}}{\mathrm{x}^2-\mathrm{c}^2+\mathrm{y}^2}.$$

Thus, the streamlines are given by $\frac{2\mathrm{cy}}{\mathrm{x}^2+\mathrm{y}^2-\mathrm{c}^2} = \tan\frac{\psi}{\mathrm{m}}$ which is rewritten as

$\mathrm{x}^2+\mathrm{y}^2-2\mathrm{cy}\cot\frac{\psi}{\mathrm{m}}-\mathrm{c}^2 = 0,$ where ψ is assume to be constant.

The above streamline represents acircle with radius $\mathrm{c}\sqrt{\cot^2\psi/\mathrm{m}+1}$ and centre $\mathrm{c}\cot\psi/\mathrm{m}$ lying on the y-axis. Each value of ψ will give a streamline.

Sources and sinks within a cylinder in a flow

47. Discuss the flow within a circular cylinder of radius a in the presence of a source and sink of strength m located at $(a/2,\ 0)$ and $(-a/2,\ 0)$.

Ans: For a source of strength m at $(a/2,\ 0)$ and sink of strength m located at $(-a/2,\ 0)$, the complex potential with no circular boundary is

$f(z) = m\ln\left(z-\frac{a}{2}\right) - m\ln\left(z+\frac{a}{2}\right)$

Hence, $\overline{f}\left(\frac{a2}{z}\right)=m\ln\left(\frac{a^2}{z}-\frac{a}{2}\right)-m\ln\left(\frac{a^2}{z}+\frac{a}{2}\right)$

Thus, when a cylinder $x^2+y^2=a^2$ is inserted into the fluid, the complex potential at points of its interior is

$$W(z)=m\left\{\ln\left(z-\frac{a}{2}\right)-\ln\left(z+\frac{a}{2}\right)+\ln\left(\frac{a^2}{z}-\frac{a}{2}\right)-\ln\left(\frac{a^2}{z}+\frac{a}{2}\right)\right\}$$

$$=-m\left\{\ln\left(x+\frac{a}{2}+iy\right)-\ln\left(x-\frac{a}{2}+iy\right)+\ln\left(2a+x+iy\right)-\ln\left(2a-x-iy\right)\right\}$$

which gives

$$\psi=-m\left\{\tan^{-1}\left(\frac{y}{x+a/2}\right)-\tan^{-1}\left(\frac{y}{x-a/2}\right)+\tan^{-1}\left(\frac{y}{2a+x}\right)+\tan^{-1}\left(\frac{y}{2a-x}\right)\right\}$$

$$\Rightarrow\psi=-m\left\{\tan^{-1}\left(\frac{\frac{y}{x-a/2}-\frac{y}{x+a/2}}{1+\frac{y^2}{x^2-a^2/4}}\right)+\tan^{-1}\left(\frac{\frac{y}{2a-x}+\frac{y}{2a+x}}{1-\frac{y^2}{4a^2-x^2}}\right)\right\}$$

$$=-m\left\{\tan^{-1}\left(\frac{4ay}{4a^2-r^2}\right)-\tan^{-1}\left(\frac{ay}{r^2-a^{2/4}}\right)\right\}$$

$$\text{or}\ \psi=m\tan^{-1}\left(\frac{5ay(a^2-r^2)}{(4a^2-r^2)(r^2-a^2/4)+4a^2y^2}\right)$$

Thus, the streamlines are given by $\psi=$ constants which yields $(4a^2-r^2)(r^2-a^2/4)+4a^2y^2=Ky(a^2-r^2)$ as the required streamlines with K being a constant.

Superposition of sources and sinks

48.If there are source located at $(a,0)$, $(-a,0)$ and sink at $(0,a)$ and $(0,-a)$ all of equal strength in flow, then show that the circles through these points is a streamline.

Ans: Let $W(z)$ be the complex potential associated with the flow having sourcesat $(a,0)$, $(-a,0)$ and sinks at $(0,a)$ and $((0,-a)$ each of strength *m*. Thus $W(z)$ is given by

$$w(z)=m\ln(z-a)+m\ln(z+a)-m\ln(z-ia)-m\ln(z+ia)$$

$$\Rightarrow\phi+\psi=m\ln(z^2-a^2)-m\ln(z^2+a^2)$$

$$=m\left\{\ln(r^2e^{2i\theta}-a^2)-\ln(r^2e^{2i\theta}+a^2)\right\}$$

$$=m\ln\frac{r^2e^{2i\theta}-a^2}{r^2e^{2i\theta}+a^2}$$

$$=m\ln\frac{(r^2\cos2\theta-a^2)+ir^2\sin2\theta}{(r^2\cos2\theta+a^2)+ir^2\sin2\theta}$$

$$\Rightarrow \psi = m\left(\tan^{-1}\frac{r^2\cos 2\theta}{r^2\cos 2\theta - a^2} - \tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}\right) \quad \left(\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right)$$

$$= m\tan^{-1}\left(\frac{2ar^2\sin 2\theta}{r^4\cos^2 2\theta - a^4 + r^4\sin^2 2\theta}\right)$$

$$= m\left(\ln(r^2e^{2i\theta} - a^2) - \ln(r^2e^{2i\theta} + a^2)\right) \qquad \text{(A)}$$

Thus, the stream lines are given by

$$\frac{2a^2r^2\sin 2\theta}{r^4 - a^4} = k \quad \text{(A constant)}$$

$$\Rightarrow \frac{r^4 - a^4}{2a^2r^2\sin 2\theta} = \frac{1}{k} = c\,\text{(Say)}$$

In particular, for $c = 0$, $r^4 = a^4 \Rightarrow r^2 = a^2$, which are circles with centre at origin and of radius a. Thus, the circle passes through the said points.

Alternately

$$w(z) = m\ln(z^2 - a^2) - m\ln(z^2 + a^2)$$

$$\Rightarrow \phi + \psi = m\ln(x^2 - y^2 + 2ixy - a^2) - m\ln(x^2 - y^2 + 2ixy + a^2)$$

Thus, $\psi = \tan^{-1}\frac{2xy}{x^2 - y^2 - a^2} - \tan^{-1}\frac{2xy}{x^2 - y^2 + a^2} == \tan^{-1}\left(\frac{\frac{2xy}{x^2-y^2-a^2} - \frac{2xy}{x^2-y^2+a^2}}{1 + \frac{2xy}{x^2-y^2-a^2}\cdot\frac{2xy}{x^2-y^2+a^2}}\right)$

$$= \tan^{-1}\frac{4a^2xy}{\left(x^2+y^2\right)^4 - a^4} \;==\tan^{-1}\frac{4a^2xy}{\left(x^2+y^2+a^2\right)\left(x^2+y^2-a^2\right)}.$$

Hence, ψ = constant

$$\Rightarrow \tan^{-1}\frac{4a^2xy}{\left(x^2+y^2+a^2\right)\left(x^2+y^2-a^2\right)} = K\,\text{(a constant)}$$

$$\Rightarrow \frac{4a^2xy}{\left(x^2+y^2+a^2\right)\left(x^2+y^2-a^2\right)} = \tan K = C \Rightarrow \frac{\left(x^2+y^2+a^2\right)\left(x^2+y^2-a^2\right)}{4a^2xy} = \frac{1}{C} = c$$

Thus, $c = 0 \Rightarrow x^2 + y^2 = a^2$ which is the equation of a circle with centre at origin and passing through the points as stated and is the required streamline.

49. In a two dimensional motion, sinks of strength *m* is placed at each of the point (-c, 0) and (c, 0) and a source of strength *2m* is placed at the origin. Find the streamlines for the flow field.

Ans: Here the complex potential is given by

$$w(z) = -m\ln(z-c) - m\ln(z-c) + 2m\ln z \; = -m\ln(z^2 - c^2) + m\ln z^2$$

Hence, $\phi + i\psi = -m\ln(x^2 - y^2 - c^2 + 2ixy) + m\ln(x^2 - y^2 + 2ixy)$

Thus, $\psi = -m\tan^{-1}\dfrac{2xy}{x^2 - y^2 - c^2} + m\tan^{-1}\dfrac{2xy}{x^2 - y^2}$

$$= -m\tan^{-1}\frac{\dfrac{2xy}{x^2-y^2-c^2} - \dfrac{2xy}{x^2-y^2}}{1 + \dfrac{2xy}{x^2-y^2-c^2}\cdot\dfrac{2xy}{x^2-y^2}} = -m\tan^{-1}\frac{2xyc^2}{(x^2-y^2)(x^2-y^2-c^2)+4x^2y^2}.$$

Thus, $\psi =$ constant gives the streamlines.

Therefore, the streamlines are given by $\dfrac{2xyc^2}{(x^2-y^2)(x^2-y^2-c^2)+4x^2y^2} = \tilde{k}$

i.e. $(x^2 - y^2)(x^2 - y^2 - c^2) + 4x^2y^2 = kc^2xy$

$\Rightarrow (x^2 - y^2)^2 - c^2(x^2 - y^2) + 4x^2y^2 = kc^2xy$

$\Rightarrow (x^2 + y^2)^2 - c^2(x^2 - y^2) = kc^2xy$

$\Rightarrow (x^2 + y^2)^2 = c^2(x^2 - y^2 + kxy)$ are the streamlines.

Further, the flow velocity is given by

$$\left|\frac{dW}{dz}\right| = \left|\frac{2mz}{z^2} - \frac{2mz}{z^2 - c^2}\right| = \frac{2mc^2}{|z||z^2 - c^2|} = \frac{2mc^2}{r_1 r_2 r_3}$$

where r_1, r_2 and r_3 are the distance of a fluid particle at any point from the sinks and source respectively.

Complex potential for the faired entry into a long parallel sided channel.

50. Show that the complex potential $z = e^w + w$ represents a faired entry to a long parallel sided channel for an incompressible fluid flow.

Ans: Given $z = e^w + w$, (A)

where $z = x + iy$, $w = \phi + i\psi$

Hence, $x + iy = \phi + i\psi + e^{\phi + i\psi}$

$= \phi + e^{\phi}\cos\psi + i(\psi + e^{\phi}\sin\psi)$ (B)

Comparing real and imaginary parts of (B), it is derived that

$x = \phi + e^{\phi}\cos\psi$ (C)

and $y = \psi + e^{\phi}\sin\psi$ (D)

which can be rewritten as

$x - \phi = e^{\phi}\cos\psi$ and $y - \psi = e^{\phi}\sin\psi$ (E)

Therefore, $\tan\psi = \dfrac{y - \psi}{x - \phi} \Rightarrow \phi = x - \dfrac{y - \psi}{\tan\psi}$ (F)

Substituting for ϕ from (F), it is derived that $y = \psi + e^{x - \frac{y-\psi}{\tan\psi}\sin\psi}$

Thus, the streamlines are obtained from (F) by assuming $\psi = \text{constant}$.

In particular, $\psi = \pm\pi/2$ yield

$y = \pi/2 + e^{x}$ (G)

and $y = -\pi/2 - e^{-x}$ (H)

Thus, Equations (G) and (H) together give the stream lines which behave like a faired entry into a long parallel sided channel.

Complex potential (irrotational flow in a convergent divergent channel-streamlines are rectangular hyperbolas)

51.Show that the complex potential $w(z) = \cosh^{-1}\frac{z}{a}$ represents the irrotational flow in a convergent and divergent channel.

Ans: Given $w(z) = \cosh^{-1}\frac{z}{a}$.

Substituting $z = x + iy,\ w = \phi + i\psi$, the complex potential $w(z)$ is rewritten as

$x + iy = c\cosh(\phi + i\psi)$

$= c\cosh\phi\cos\psi + ic\ \sinh\phi\sin\psi$

which yield $x = c\cosh\phi\cos\psi,\ y = c\sinh\phi\sin\psi$.

Given the identity $\cosh^2\phi - \sinh^2\phi = 1,$ (C) yields

$$\left(\frac{x}{c\cos\psi}\right)^2 - \left(\frac{y}{c\sin\psi}\right)^2 = 1$$

Now, $\psi = \text{constant}$ yield that the streamlines are rectangular hyperbolas which are convergent and divergent channels in a flow.Further using the identity $\cos^2\psi + \sin^2\psi = 1,$ it can be derived that $\left(\frac{x}{c\cosh\phi}\right)^2 + \left(\frac{y}{c\sinh\phi}\right)^2 = 1$.

Thus, $\phi = \text{constant}$ yield the equipotential surfaces which are given by

$$\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 = 1$$

where α and β are constants given by $\alpha = c\cos\phi$ and $\beta = c\sin\phi$.

Complex potential for which streamlines are confocal ellipses

52. Discuss the flow whose complex velocity potential is given by $z = c\cos w$, where c is a constant.

Ans: Given $z = c\cos w$

$\Rightarrow x + iy = c\cos(\phi + i\psi)$

$\Rightarrow x = c\cosh\psi\cos\phi,\;\; y = \text{-}c\sinh\psi\sin\phi$

Eliminating ϕ,

$$\frac{x^2}{c^2\cosh^2\psi} + \frac{y^2}{c^2\sinh^2\psi} = 1$$

Thus, for $\psi = k$ (a constant) yield

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$

where $A = c\cosh\psi,\;\; B = c\sin\psi$ are constants. Thus, the streamlines are confocal ellipses whose semi-axes are *A* and *B*.

Further, $\frac{1}{q^2} = \frac{dz}{dw}\cdot\frac{dw}{d\bar{z}} = \sqrt{c^2 - z^2}\sqrt{c^2 - \bar{z}^2}$ which vanishes when $z = \pm c$ which shows the flow speed is infinity at $z = \pm c$.

Complex potential (circular cylinder in an infinite fluid which is at rest at infinity)

53. Discuss the motion of a circular cylinder of radius a moving with velocity u along the x- axis in an infinite fluid which is at rest at infinity.

Ans. Assuming the flow as irrotational, there exist a velocity potential $\phi(x, y)$ which satisfies the Laplace equation. Since the circular cylinder is moving with velocity u along the x- axis which is at rest at infinity, the velocity potential $\phi(x, y)$ satisfies the boundary conditions given by

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = -u\cos\theta,\;\; \left.\frac{\partial\phi}{\partial r}\right|_{r=\infty} = 0 \text{ and } \left.\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{r=\infty} = 0.$$

Using the condition at $r = a$, the velocity potential can be written in the form

$\phi(r,\theta) = Ar\cos\theta + \frac{B}{r}\cos\theta,$ which using the wall boundary condition yield $A - \frac{B}{a^2} = u.$ Next, using the condition at infinity, it is derived that $A = 0,\; B = ua^2.$ Thus, the velocity potential is obtained as $\phi = \frac{ua^2\cos\theta}{r}$. Next, to obtain the stream function for the flow, we will use the two relations $\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$.

Thus, the stream function $\psi(r,\theta)$ is obtained as $\psi(r,\theta) = -\frac{ua^2\sin\theta}{r}$. Therefore, the complex potential for the flow is obtained as $w(z) = \phi + i\psi = \frac{ua^2}{r}\left(\cos\theta - i\sin\theta\right) = \frac{ua^2}{z}$.

Complex potential (Flow past a circular cylinder in the presence of circulation, application of Blasius theorem)

54. Find the force and moment exerted on a circular cylinder of radius a centre at origin in a uniform stream of speed u with circulation Γ.

Ans: The complex potential for the uniform flow past acylinder of radius a with circulation Γ is given by

$$w(z) = u\left(z + \frac{a^2}{z}\right) + i\Gamma \ln z \Rightarrow \frac{dw}{dz} = u\left(1 - \frac{a^2}{z^2}\right) + \frac{i\Gamma}{z}$$

Hence, by Blasius theorem

$$Y + iX = -\frac{\rho}{2}\oint_c \left(\frac{dw}{dz}\right)^2 dz = -\frac{\rho}{2}\oint \left\{u + \frac{i\Gamma}{z} - \frac{ua^2}{z^2}\right\}^2 dz$$

$$= -\frac{\rho}{2} 2\pi i \left(\text{Residue of integrand at } z = 0\right) = -\frac{\rho}{2} 2\pi i (2iu\Gamma) = 2\pi\rho\Gamma \text{u}.$$

Further, the moment M is given by

$$M = -\frac{\rho}{2} \text{Re} \oint z\left(\frac{dw}{dz}\right) dz = 0.$$

Complex potential (Uniform flow past a circular cylinder in the presence of circulation)

55. Find the stagnation point associated with the flow field for which the complex potential is given by $w(z) = u\left(z + a^2/z\right) + i\Gamma \ln z.$

Ans: The complex potential for a flow field is given by

$$w = uz + \frac{ua^2}{z} + i\Gamma \ln z$$

Thus, in the polar co-ordinate system, the complex potential $w(z)$ yields

$$\phi = u\left(r + \frac{a^2}{r}\right)\cos\theta - \Gamma\theta \Rightarrow q_r = \frac{\partial \phi}{\partial r} = u\left(1 - \frac{a^2}{r^2}\right)\cos\theta,\ q_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -u\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{r}.$$

Thus, $$q\Big|_{r=a} = \left\{\left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2\right\}_{r=a}^{1/2} = 2u\sin\theta + \frac{\Gamma}{r}.$$

Near a stagnation point speed vanishes, which yields $2u\sin\theta + \Gamma = 0$. Therefore, when there no circulation, $\Gamma = 0$ which gives that stagnation points are for $\theta = 0, \pi$. On the other hand, in the presence of circulation, the stagnation points are given by

$$\sin\theta = \frac{-\Gamma}{2ua}$$

which is possible only when $|\sin\theta| \le 1$ or $|\Gamma| \le 2au.$

Case 1: For $|\Gamma| < 2au \Rightarrow |\sin\theta| < 1$.

Here, the stagnation point lies on the cylinder on a line below the centre parallel to x-axis.

Case 2: $|\Gamma| = 2au$, in this case $\sin\theta = -1$. Here, the stagnation point coincides at the bottom of the cylinder.

Case 3: If $|\Gamma| > 2au,$ then there is no stagnation point on the cylinder. However, the stagnation point will lie above the cylinder for $\Gamma > 0$ and below the cylinder for $\Gamma < 0$.

Superposition of complex potentials does not represent individual flow patterns

56. Prove that the complex potential $W(z) = u\left(z + a^2/z\right) - m\ln(z - z_0)$ does not represent the flow past a cylinder in the presence of the sink.

Ans: Given $W(z) = u\left(z + a^2/z\right) - m\ln(z - z_0) \Rightarrow \dfrac{dW}{dz} = u\left(1 - a^2/z^2\right) - \dfrac{m}{z - z_0}$

It is obvious that at $z = z_0$, the flow does not represent a uniform flow past a cylinder. Further, the flow speed should be $q = 2u\sin\theta$ on the boundary of the cylinder at $|z| = a$ in case of uniform flow past a cylinder which is not the case here.

57. Show that the complex potential $W(z) = u\left(z + a^2/z\right) + m\ln(z - z_0)$ does not represent the uniform flow past a cylinder of radius a in the presence of a source of strength m located at $z = z_0$.

Ans: On the circle $|z| = a$, the stream function for the flow is not a constant which ensures the cylinder is not a stream line.

Note: Criteria for additive property of sources in an uniform flow: When no boundaries occur in the fluid, the motion due to an uniform flow, any number of source can be obtained by addition of the corresponding complex potentials.

Complex potentials for vortex of strength k.

58. Describe the irrotational motion of an incompressible fluid whose complex potential is given by $w(z) = ik\ln z$.

Ans: Given $w(z) = ik\ln z = ik\ln re^{i\theta} = ik\ln r - k\theta$

$\Rightarrow \phi = -k\theta, \psi = k\ln r$

$\Rightarrow \psi =$ constant yields $r =$ constant, which says that streamlines are concentric circles with canter at origin. On the other hand, $\theta =$ constant yields lines through origin cutting the circles orthogonally.

Complex potentials for irrotational flow produced by a line vortex

59. Show that the complex potential $w(z) = -\left(i\Gamma/2\pi\right)\ln z$ yield a singular distribution of vorticity $\vec{\Omega} = \Gamma\delta(x)$ concentrating at origin.

Ans: $w(z) = -\left(i\Gamma/2\pi\right)\ln z$

$$\Rightarrow \phi = \frac{\Gamma\theta}{2\pi}, \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$$

Writing $\psi = \lim_{\in\to 0}\frac{-\Gamma}{4\pi}\ln(r^2+\in^2) \Rightarrow \nabla^2\psi = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\psi = \lim_{\in\to 0}\frac{-\Gamma\in^2}{\pi(r^2+\in^2)^2} = -\Gamma\delta(x),$ yields $\vec{\Omega} = \Gamma\delta(x)$ which shows that there is a singular distribution of vorticity $\vec{\Omega} = \Gamma\delta(x)$ concentrated at the origin.The complex potential $W(z)$ describes the irrotational flow produced by a line vortex of strength Γ concentrated at the origin.

60. The complex velocity potential $W(z) = \left(i\Gamma/2\pi\right)\ln z$ describes the irrotational flow produced bya line vortex of strength Γ located at $z = 0$.

Ans: Given $W(z) = \frac{i\Gamma}{2\pi}\ln z \Rightarrow \phi + i\psi = \frac{i\Gamma}{2\pi}\ln re^{i\theta} = \frac{i\Gamma}{2\pi}\ln r - im\theta$

$$\Rightarrow \psi = \frac{\Gamma}{2\pi}\ln r = \lim_{\in\to 0}\frac{\Gamma}{4\pi}\ln(r^2+\in^2)$$

Thus, $\nabla^2\psi = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\psi = \lim_{\in\to 0}\frac{\Gamma\in^2}{\pi(r^2+\in^2)^2} = \Gamma\delta(x)$

Hence, $\vec{\Omega} = \Gamma\delta(x)\hat{k}$, which says that the vorticity is $\Omega = \Gamma\delta(x)$ concentrated at the origin. Here $\psi =$ constant yields, $r =$ constant, which says the stream lines are circles with center at origin and the flow speed is $q_r = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$.

Application of Joukowski Transformation (Uniform flow past an elliptic cylinder)

61. Using the circle theorem, find the uniform flow past an elliptic cylinder with a and b being the semi-major and semi-minor axes and centre being at the origin.

Ans: In the z-plane, the complex potential for uniform flow past a circular cylinder of radius $\left(a+b\right)/2$ is given by

$$w(z) = U\left(z + \frac{(a+b)^2}{4z}\right) \qquad \text{(A)}$$

Now, the transformation

$$Z = \frac{1}{2}\left(z + \sqrt{z^2 - c^2}\right), \text{ where } c^2 = a^2 - b^2 \qquad \text{(B)}$$

maps the region outside an ellipse with a, b being the major and minor axes in the z-plane on the region outside a circle with centre being at origin and radius $r = (a+b)/2.$ in the z-plane. Thus, putting for z from (B) in (A), it is derived that

$$w(z) = \frac{u}{2}\left\{z + \sqrt{z^2 - c^2} + \frac{(a+b)^2}{4(z + \sqrt{z^2 - c^2})}\right\} = \frac{u}{2}\left\{z + \sqrt{z^2 - c^2} + \frac{(a+b)^2 (z + \sqrt{z^2 - c^2})}{4c^2}\right\}$$

$$= \frac{u(a+b)}{2}\left\{\frac{z + \sqrt{z^2 - c^2}}{a+b} + \frac{z - \sqrt{z^2 - c^2}}{a-b}\right\}.$$

Now substituting $z = c\cosh\zeta$ and using the results

$$z + \sqrt{z^2 - c^2} = c\cosh\zeta + c\sinh\zeta = ce^{i\zeta},\ z - \sqrt{z^2 - c^2} = c\cosh\zeta - c\sinh\zeta = ce^{-i\zeta},$$

$$\Rightarrow w(z) = \{u(a+b)/2\}\left\{e^{i(\zeta-\zeta_0)} + e^{-i(\zeta-\zeta_0)}\right\} = u(a+b)\cosh(\zeta - \zeta_0).$$

Hence, $w(z) = u(a+b)\cosh(\zeta - \zeta_0)$, where $z = c\cosh\zeta$, $\zeta_0 = \frac{1}{2}\ln\left(\frac{a+b}{a-b}\right)$ is the uniform flow past an elliptic cylinder.

Application of Blasius theorem to uniform flow past an elliptic cylinder

62. Find the force and moment due to the uniform flow past an elliptic cylinder.

Ans: The complex potential associated with the uniform flow past an elliptic cylinder is given by $w(z) = u(a+b)\cosh(\zeta - \zeta_0)$

where $z = c\cosh\zeta$, $\zeta_0 = \frac{1}{2}\ln\left(\frac{a+b}{a-b}\right)$, $c^2 = a^2 - b^2$

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz} = \frac{dw}{d\zeta}\bigg/\frac{dz}{d\zeta} = u(a+b)\frac{\sinh(\zeta - \zeta_0)}{c\sinh\zeta}$$

$$= \frac{u(a+b)}{c}\left\{\cosh\zeta_0 - \sinh\zeta_0 \frac{z}{\sqrt{z^2 - c^2}}\right\}.$$

Now $\dfrac{z}{\sqrt{z^2 - c^2}} = 1 + \dfrac{c^2}{2z^2} + \ldots$

Therefore, $\dfrac{dw}{dz} = \dfrac{u(a+b)}{c}\left\{e^{-\zeta_0} - \dfrac{c^2\sinh\zeta_0}{2z^2} + \ldots\right\} \Rightarrow \left(\dfrac{dw}{dz}\right)^2 = A^2\left(e^{-2\zeta_0} - \dfrac{c^2 e^{-\zeta_0}\sinh\zeta_0}{z^2} + \ldots\right).$

Thus, by Blasius theorem, $X - iY = \dfrac{1}{2}i\rho\oint_c \left(\dfrac{dw}{dz}\right)^2 dz = 0.$

Further, $M + iN = \text{Real part} -\frac{\rho}{2}\oint_c \left(\frac{dw}{dz}\right)^2 dz = -\frac{\rho}{2} 2\pi i \left\{ -c^2 \frac{u^2(a^2+b^2)^2 e^{-\zeta_0} \sinh\zeta_0}{c^2} \right\}$

$$= \frac{\rho\pi i}{2}\left(\frac{u(a+b)}{c}\right)^2 \left\{1 - e^{-2\zeta_0}\right\}.$$

Writing $\zeta_0 = \xi_0 + i\eta_0$

$$M + iN = \frac{\rho\pi i}{2}\left(\frac{u(a+b)}{c}\right)^2 \left\{1 - e^{-2\zeta_0}\right\}(\cos 2\zeta_0 - \cos 2\zeta_0)$$

Therefore, $M = \rho\pi \left\{\frac{u(a+b)}{c}\right\}^2 e^{-2\zeta_0} \sin\eta_0 \cos\eta_0$ (C)

We have $\zeta_0 = \xi_0 - i\eta_0 = \frac{1}{2}\ln\left(\frac{a+b}{a-b}\right)$

Further, $\frac{u^2(a+b)^2}{c^2} = u^2\left(\frac{a+b}{a-b}\right) = u^2 e^{2i\zeta_0}$ (D)

Hence from (C) and (D) $M = -\pi\rho(a^2 - b^2)u^2 \sin\alpha\cos\alpha.$

Application of conformal mapping

63. Discuss the flow associated with the complex potential given by $w = -iu/Z^2$.

Ans. Writing $Z = \frac{1}{2}\left(z + \sqrt{z^2 - c^2}\right), \qquad c^2 = a^2 - b^2.$

$$\text{Thus, } w(z) = -\frac{4iu}{\left(z + \sqrt{z^2-c^2}\right)^2} = -\frac{4iu\left(z - \sqrt{z^2-c^2}\right)^2}{\left(z + \sqrt{z^2-c^2}\right)^2}$$

$$= \frac{-4iu\left(z^2 + z^2 - c^2 + 2z\sqrt{z^2-c^2}\right)}{\left(a^2-b^2\right)^2} = \frac{-4iu\left(c^2 e^{-2i\zeta}\right)}{\left(a^2-b^2\right)^2} = \frac{-4iue^{-2i\zeta}}{c^2}.$$

Result: The streamlines associated with the general motion of a cylinder in two dimensions is given by $\psi(x,y) = ux - vy + \frac{w}{2}(x^2 + y^2) + c$ where (u,v) are the components of linear velocity and w is the angular velocity.

Complex potential (Elliptic cylinder rotating in an infinite mass of fluid)

64. Find the complex potential and stream function associated with an elliptic cylinder rotating in an infinite mass of liquid which is at rest at infinity.

Ans: Let the cross-section of the elliptic cylinder be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where a and b are the semi-major and semi-minor axes. Let $\zeta=\alpha$ be the boundary of the cylinder. Then the transformation

$z=c\cosh\zeta$ yields $a=c\cosh\alpha,\ b=c\sinh\alpha.$

Since the cylinder is rotating with angular velocity ω, the general motion is of the form

$$\psi=\frac{\omega}{2}(x^2+y^2)+B, \qquad \text{(A)}$$

where B is the constant to be determined. Further, assuming

$$W(z)=-iAz^2 \Rightarrow \phi+i\psi=-iA(x+iy)^2=-iA(x^2-y^2+2ixy)$$

$$\Rightarrow \phi=2Axy, \quad \psi=-A(x^2-y^2) \qquad \text{(B)}$$

Comparing (A) and (B), it is derived that

$$-A(x^2-y^2)=\frac{\omega}{2}(x^2+y^2)+B$$

which is rewritten as $\frac{x^2}{-B\left(A+\frac{\omega}{2}\right)}+\frac{y^2}{B\left(A-\frac{\omega}{2}\right)}=1$

and is equivalent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Thus, $a^2=\frac{-B}{\frac{1}{2}\omega+A},\ b^2=\frac{-B}{\frac{1}{2}\omega-A} \Rightarrow \frac{\omega}{2}(a^2-b^2)=-A(a^2+b^2)$

Hence, from (B) it is derived that

$$\phi=-\omega\left(\frac{a^2-b^2}{a^2+b^2}\right)xy,\ \psi=\frac{\omega}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)(x^2-y^2).$$

Result:Let the equation of boundary of the cross-section of a cylinder containing liquid is given by $z\bar{z}=f(z)+\bar{f}(\bar{z})$ where $f'(z)$ has no singularities within the cross-section. Then the complex potential associated the cylinder containing the fluid which rotates about an axis through the origin parallel to the generators is given by $w(z)=i\omega\bar{f}(z)$. Then, $\psi=\omega z\bar{z}/2$ on the boundary.

Complex potential (Elliptic cylinder rotating in an infinite mass of fluid)

65. Using the above result, show that $w(z)=\frac{i\omega}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)z^2$ is the complex potential associated with the elliptic cylinder which rotates about an axis through the origin and is parallel to the generator.

Ans. The equation of an ellipse is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which can be rewritten as

$$\frac{(z+\overline{z})^2}{4a^2}-\frac{(z-\overline{z})^2}{4b^2}=1 \text{where } x=\frac{z+\overline{z}}{2},\ \ y=\frac{z-\overline{z}}{2i}.$$

Thus, the equation of the ellipse is rewritten as

$$b^2\left\{z^2+\overline{z}^2+2z\overline{z}\right\}-a2\left\{z^2+\overline{z}^2-2z\overline{z}\right\}=4a^2b^2$$

$$\Rightarrow (b^2-a^2)z^2+(b^2-a^2)\overline{z}^2+2z\overline{z}(b^2+a^2)=4a^2b^2$$

$$\Rightarrow z\overline{z}=\frac{a^2b^2}{a^2+b^2}+(a^2-b^2)z^2+(a^2-b^2)\overline{z}^2 \ \ =(a^2-b^2)z^2+\frac{a^2b^2}{a^2+b^2}+(a^2-b^2)\overline{z}^2+\frac{a^2b^2}{a^2+b^2}$$

$$\Rightarrow z\overline{z} \ =f(z)+\overline{f}(\overline{z}),$$

where $\quad f(z)=\frac{1}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)z^2+\frac{a^2b^2}{a^2+b^2}.$

Thus, $w(z)=i\omega f(z)=i\omega\left\{\frac{1}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)z^2+\frac{a^2b^2}{a^2+b^2}\right\}=\frac{i\omega}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)z^2+\frac{a^2b^2(i\omega)}{a^2+b^2}$ is the complex potential for flow inside the cylinder where the constant term may be taken as zero without loss of generality. Therefore $w(z)=\frac{i\omega}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)z^2$ is the required complex potential.

Potential flow between two concentric cylinders

66. Find the velocity potentials associated with the flow between two concentric cylinders when the inner cylinder is moved suddenly with velocity u perpendicular to the axis of the cylinder with the outer one being kept fixed.

Ans. Assuming that the flow is irrotational within the cylinders, the corresponding velocity potential $\phi(r,\theta)$ satisfies the Laplace equation. Assuming that the inner cylinder is of radius a and outer cylinder is of radius b, the boundary conditions on the cylinders are given by

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a}=-u\cos\theta,\ \ \left.\frac{\partial\phi}{\partial r}\right|_{r=b}=0.$$

Here, the velocity potentials are assumed to be of the form

$$\phi(r,\theta)=\left(Ar+\frac{B}{r}\right)\cos\theta+\left(Cr+\frac{D}{r}\right)\sin\theta$$

Using the boundary conditions, it is derived that

$$\left(A-\frac{B}{a^2}\right)\cos\theta+\left(C-\frac{D}{a^2}\right)\sin\theta=-u\cos\theta$$

and $$\left(A-\frac{B}{b^2}\right)\cos\theta+\left(C-\frac{D}{b^2}\right)\sin\theta=0$$

which yield $\left(A-\frac{B}{a^2}\right)=-U,\left(A-\frac{B}{b^2}\right)=0,\left(C-\frac{D}{a^2}\right)=0,\ \left(C-\frac{D}{b^2}\right)=0.$

Solving the above set of equations for A, B, C, D it is derived that

$$A=\frac{Ua^2}{b^2-a^2},B=\frac{Ua^2b^2}{b^2-a^2}\text{ and }C=D=0.$$

Thus, $\phi(r,\theta)=\frac{Ua^2}{b^2-a^2}\left(r+\frac{b^2}{r}\right)\cos\theta.$

Further, it can be easily derived that the stream function for the flow is given by

$$\psi(r,\theta)=\frac{-Ua^2}{b^2-a^2}\left(r+\frac{b^2}{r}\right)\sin\theta.$$

Plane progressive gravity waves

67. The surface profile associated with the motion of a plane gravity wave is given by $\eta(x,t)=a\sin(kx-\omega t)$. Determine the velocity potential $\phi(x,y,t)$.

Ans: In case of surface gravity wave, the velocity potential $\phi(x,y,t)$ satisfies

$$\nabla^2\phi=0, \qquad (A)$$

subject to the boundary conditions

$$\frac{\partial\phi}{\partial t}+g\eta=0 \text{ on } y=0, \qquad (B)$$

and the kinematic condition

$$\eta_t=\phi_y \text{ on } y=0. \qquad (C)$$

Further, the bottom boundary condition is given by

$$\frac{\partial\phi}{\partial y}=0 \text{ on } y=-h. \qquad (D)$$

Since $\eta(x,t)=a\sin(kx-\omega t)$, the kinematic /dynamic boundary conditions ensure that ϕ must be a cosine function. Thus, a separable solution can be assumed to be of the form

$$\phi(x,y,t)=f(y)\cos(kx-\omega t), \qquad (E)$$

where $f(y)$ to be determined.

Substituting for ϕ in equation (A), we obtain f satisfies

$$\frac{d^2f}{dy^2}-k^2f=0, \qquad (F)$$

whose solution is of the form

$$f(y) = A\cosh k(h+y) + B\sinh k(h+y) \qquad \text{(G)}$$

with A, B being arbitrary constants. Substituting for $f(y)$ from (G) in (E), condition (D) yields

$$\frac{df}{dy} = 0 \text{ on } y = -h,$$

which yields $B = 0$.

Thus, $\phi(x, y, t)$ is of the form $\phi(x, y, t) = A\cosh k(h+y)\cos(kx - \omega t)$ (H)

Substituting for ϕ in (B), it is derived that

$$A\omega\cosh kh + ga = 0 \Rightarrow A = \frac{-ga}{\omega\cosh kh}$$

Further, from condition (C), it is derived that

$$\omega^2 = gk\tanh kh \qquad \text{(I)}$$

Therefore, the velocity potential ϕ for $\eta = a\cos(kx - \omega t)$ is given by

$\phi = -\dfrac{ag}{\omega}\dfrac{\cosh k(h+y)}{\cosh kh}\cos(kx - \omega t)$, where k satisfies the dispersion relation given in (I).

Examples on plane waves

68. Find the wave length in terms of wave period in deep water under the small amplitude theory and thus find the wave length for a 10s period in deep water.

Ans: From dispersion relation

$\omega^2 = gk$ (In case of deep water)

$$\left(\frac{2\pi}{T}\right)^2 = g\frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{gT^2}{2\pi} = 1.56T^2\text{m} = 5.12T^2\text{ft.}$$

Therefore, $c = \dfrac{\lambda}{T} = 1.56T\text{m/s} = 5.12\text{ft/sec.}$

Thus, in case of a 10sec. period wave in deep water, $\lambda = 1.56(100)\text{m} = 156\text{m} = 512\text{ft.}$

69. A plane progressive wave is propagating of depth 100m having a period of 10s and height of 2m. Find the wave celerity and wave steepness.

Ans: Assume it is a case of deep water wave.

Thus $\lambda = 1.56T^2 = 156\text{m}$

Here $\dfrac{h}{\lambda} = \dfrac{100}{156} > \dfrac{1}{2}$

Thus, the deep water wave assumption is justified.

Now $c = 1.56T = 15.6\text{m/s}$

Wave steepness $\dfrac{H}{\lambda} = \dfrac{2}{156} = 0.013$

70. A wave in water 120m deep has a period of 8s and height of 4m. Determine the wave celerity,wavelength.

Ans: Assume deep water wave.

Thus, the dispersion relation gives

$\lambda = 1.56(8)^2 \text{m} = 99.82\text{m}$

Therefore, wave celerity $c = \dfrac{\lambda}{T} = 1.56 \times 8\text{m/s} = 12.4775\text{m/s}$

Here $\dfrac{h}{\lambda} = \dfrac{126}{99.82} > \dfrac{1}{2}$, so the deep water assumption is justified.

71. A plane gravity wave is propagating in an infinitely extended channel of average depth 2.3m and has a period of 10s.

Ans: Assumed that it is a case of shallow water.

Therefore $c = \sqrt{gh} = \sqrt{9.81 \times 2.3} = 4.75\text{m/s}$

and $\lambda = cT = 4.75 \times 10\text{m=}47.5\text{m}$

Now $\dfrac{h}{\lambda} = \dfrac{2.3}{47.5} = 0.048 < 0.05$

Hence, the assumption of shallow water wave is justified.

72. A plane progressive wave is propagating from deep sea normally towards the shore with straight and parallel contours. In deep sea, the wave length and wave height are given by 300m and 2m respectively. Find the wave length, wave height and group velocity at a depth of 30m near the shore line.

Ans: Given in a deep water $\lambda_1 = 300\text{m}, H_1 = 2\text{m}$

Since in deep water, $\lambda = 1.56T^2 \Rightarrow T = \sqrt{\lambda / 1.56} = \sqrt{300/1.56} = 13.868\text{sec.}$

While travelling from deep sea to shallow sea the wave period remains the same. Now, in water of depth 30m, assuming shallow water

$c = \sqrt{gh_2} \Rightarrow \dfrac{\lambda_2}{T} = \sqrt{gh_2}$

$\Rightarrow \lambda_2 = T\sqrt{gh_2} = 13.868\sqrt{9.8 \times 30} = 240.2\text{m}$

From law of conservation of energy flux,

$Ec_g = \text{constant}$

$\Rightarrow E_1 c_{g1} = E_2 c_{g2}$

$\Rightarrow c_g = \dfrac{c}{2}\left\{1 + \dfrac{2kh}{\sinh 2kh}\right\}$

In case of deep water,

$c_g = c/2 = \lambda/2 = 300/13.868 = 10.912$

On the other hand, in case of shallow water

$c_g = c = \sqrt{gh} = \sqrt{9.8 \times 30} = 17.155\text{m/s}$

Thus, $\lambda = cT = 17.155 \times 13.86 = 237.771\text{m}$

It is not a case of shallow water as $h/\lambda = 30/337.771 = 0.126 > 0.05$

Further, assuming deep water

$\lambda = 1.56T^2 = 300.079$

Therefore, $h/\lambda = 30/300.079 = 0.009 < 1/2$

Hence, it is not the case of deep water.

Thus, to find k, we have to solve

$\omega^2 = gk \tanh kh$

It can be checked that $k = 0.03 \Rightarrow \lambda_2 = 2\pi/k = 209.33\text{m}$

Therefore, $c_2 = \lambda/T = 209.33/13.86 = 15.103\text{m/s}$

and $c_{g2} = \frac{c}{2}\left\{1 + \frac{2kh}{\sinh 2kh}\right\} = \frac{15.103}{2}(1 + 0.61179) = 12.17\text{m/s}$

Therefore, from conservation of energy flux

$H_1{}^2 c_{g1} = H_2{}^2 c_{g2}$

$2^2 \times 10.822 = H_2{}^2 \times 12.17$

$\Rightarrow H_2 = 1.886\text{m}$

Therefore, wave height at depth of 30m is 1.866m.

73. A plane progressive wave in water of 100m deep has a period of 10s and a height of 2m, when it is propagated into water of depth 10m without refracting. Assuming energy losses and gained are ignored, determine the wave height and water particle velocity and pressure at a point 1m below the still water level under the wave crest.

Ans: Assume the wave propagating in deep water.

Thus, $\lambda = 1.56T^2\text{m} = 156\text{m}$

Now $h/\lambda = 100/156 > 1/2$

Thus, deep water assumption is justified when water depth $h = 10\text{m}$, $T = 10\text{s}$, from the dispersion relation $\omega^2 = gk \tanh kh$ yield

$$\lambda = \frac{gT^2}{2\pi} \tanh \frac{2\pi h}{\lambda}$$

Whose is solved to obtain

$\lambda = 93.3\text{m} \Rightarrow k = \dfrac{2\pi}{\lambda} = 0.0673$

Now $n = \dfrac{1}{2}\left\{1 + \dfrac{2kh}{\sinh 2kh}\right\} = 0.873$

Now from law of conservation of energy then

$H = H_0(c_{g0} / c_g) = 1.97\text{m}$

where subscript zero refers to waves in 100m deep. At appoint 1m below the still water level under the crest $\cos(kx - \omega t) = 1$ and $y = 1$. Hence the hydrodynamic pressure at $y = 1$, under the crest is given by

$$P = -\rho g y + \frac{\rho g H}{2}\left\{\frac{\cosh k(y+h)}{\cosh kh}\right\}\cos(kx - \omega t)$$

$$= -(100)(9.80)(-1) + \frac{1000 \times 98 \times 1.97 \times \cosh(0.0673 \times 9)}{2\cosh(0.0673) \times 10} = 19{,}113\text{N/m}^2$$

Further, water particle speed at a point 1m below the crest is

$$u = \frac{\pi H}{T}\frac{\cosh k(h+y)}{\sinh kh} = 1.01\text{m/s}$$

74. A wave in water of 2.3 deep has a period of 10s and a height of 2m. Calculate the wave speed and wavelength.

Ans: Assuming that a shallow water wave is propagating.

Thus $c = \sqrt{gh} = \sqrt{9.81 \times 2.3} = 4.75\text{m/s}$

and $\lambda = cT = 4.75 \times 10\text{m} = 47.5\text{m}$

Now $\dfrac{h}{\lambda} = \dfrac{2.3}{47.5} = 0.048 < 0.5$

Thus, the shallow water assumption is justified.

75. A wave in water of 100m deep has a period of 10s and height of 2m. Determine the wave celerity, length and steepness.

Ans: Assume deep water wave is propagating.

Thus $\lambda = cT = 156\text{m} \Rightarrow c = \dfrac{\lambda}{T} = 1.56T = 15.6\text{m/s}$

Now wave steepness $= \dfrac{H}{\lambda} = \dfrac{2}{156} = 0.013$.

76. A tsunami wave is propagating whose period 15min. and height is 0.6m at a depth of 3800m Determine the speed of propagation of the wave along with the wavelength.

Ans: Assume the case of shallow water waves.

$c = \sqrt{gh} = \sqrt{9.81 \times 3800} = 193\text{m/s (695km/hr)}$

$\lambda = cT = 193 \times 15 \times 60\text{m} = 1,73,700\text{m}$

Now $\dfrac{h}{\lambda} = \dfrac{3800}{1,73,700} = 0.021 < 0.05.$

Thus, the shallow water wave assumption is justified**.**

77. Determine the dispersion relation for a plane progressive wave propagating at the interface of two superposed fluids, which are bounded above and below, assuming that the upper fluid of density ρ_1 and depth h_1, whilst the lower fluid is of density ρ_2 and depth h_2.

Ans: Let $\eta = a\cos(kx - \omega t)$ is the fluid interface and $y = 0$ is the mean interface.

Thus, the velocity potentials for the upper and lower layer fluids are the forms

$$\phi_1 = A\cosh k(h_1 + y)\sin(kx - \omega t), \qquad -h_1 < y < 0 \qquad (1)$$

$$\phi_2 = A\cosh k(h_2 + y)\sin(kx - \omega t), \qquad -h_2 < y < 0 \qquad (2)$$

Using the continuity of velocity and pressure at the interface is same, the linearized interface conditions are given by

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + g\zeta\right) = \rho_2\left(\frac{\partial \phi_2}{\partial t} + g\zeta\right) \qquad \text{on } y = 0, \qquad (3)$$

and $$\frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \qquad \text{on } y = 0 \qquad (4)$$

From (3) and (4),

$$\rho_1\left(\frac{\partial^2 \phi_1}{\partial t^2} + g\phi_{1y}\right) = \rho_2\left(\frac{\partial^2 \phi_2}{\partial t^2} + g\phi_{2y}\right) \qquad \text{on } y = 0$$

Substituting for ϕ_1 and ϕ_2 in (4), it can be easily derived that

$$A\sin kh_1 = -B\sin kh_2$$

and $$\omega^2 = \frac{gk(\rho_2 - \rho_1)}{\rho_1 \coth kh_1 + \rho_2 \coth kh_2} = \frac{gk(1-s)}{s\coth kh_1 + \coth kh_2}$$

Now in case of deep water, $kh_1 >> 1, kh_2 >> 1$, which yield

$$\omega^2 = \frac{gk(1-s)}{1+s}$$

Further, in case of deep water, $kh_1 << 1, kh_2 << 1$, which yield

$$\omega^2 = \frac{gk(1-s)h_1h_2}{h_1^2 + sh_2}$$

78. Determine the wave dispersion relation for wave propagating in an infinitely extended channel in a two-layer fluid of density ρ_1 and ρ_2 having a free surface and an interface. The upper layer is assumed to be of depth h and lower layer is of infinite depth.

Ans: The velocity potential ϕ satisfies $\nabla^2\phi$ in the fluid region along with bottom boundary condition.

Let $z=0$ be the mean interface and $z=h$ be the free surface.

On the mean free surface, ϕ_1 satisfies

$$\frac{\partial^2\phi_1}{\partial t^2}+g\frac{\partial\phi_1}{\partial y}=0 \quad \text{at} \quad y=h$$

On the mean interface,

$$\frac{\partial\phi_1}{\partial y}=\frac{\partial\phi_2}{\partial y}$$

and $s\left(\frac{\partial^2\phi_1}{\partial t^2}+g\phi_{1y}\right)=\frac{\partial^2\phi_2}{\partial t^2}+g\phi_{2y} \quad \text{on} \quad y=0$

The velocity potentials ϕ_1 and ϕ_2 satisfies the Laplace equation and bottom boundary conditions

$\phi_1 \text{grad}\phi \to 0$ as $y\to\infty$

It can be easily derived that

$$\phi_2 = Ae^{ky}\cos(kx-\omega t), \qquad -\infty<y<0$$

$$\phi_1=\left(Be^{-ky}+Ce^{ky}\right)\cos(kx-\omega t), \qquad 0<y<h$$

Using the conditions at free surface and the interface, it is derived that

$A=C-B$ and $\omega^2=gk, \quad \omega^2=\frac{gk(\rho_1-\rho_2)(1-e^{-2kh})}{(\rho_1+\rho_2)+(\rho_1+\rho_2)e^{-2kh}}=\frac{gk1-s)}{s+\coth kh}$.

One dimensional standing waves in a channel (small amplitude gravity waves)

79. Consider the propagation of standing wave in a one dimensional channel of uniform depth h and length L. Assuming the wave profile is of the form $\eta=a\cos kx\cos\omega t$. Find the corresponding velocity potential. Thus discuss the various wave modes and the general nature of the wave profile and velocity potential.

Ans: Assuming the small amplitude approximation, the velocity potential $\phi(x,y,t)$ satisfies $\nabla^2=0$, subject to the surface boundary condition is of the form

$\phi_t+g\eta=0$ on $y=0$ and $\phi_y+\eta_t=0$ on $y=0$

Further, the bottom boundary condition is given by

$\phi_y=0$ on $y=-h.$

Further, assuming the channel is extended along x-axis from $x=0$ to $x=L$, no flow condition on the wall yields

$$\frac{\partial\phi}{\partial x}=0 \quad \text{on } x=0,L.$$

As in case of Example 1, here the velocity potential $\phi(x, y, t)$ satisfies Eq. (A) and boundary condition (C) is obtained in the form

$\phi(x, y, t) = A\cosh k(h + y)\cos kx \sin \omega t$

Since, ϕ satisfies boundary conditions (D),

so, $k = \dfrac{n\pi}{L}, \quad n = 1, 2, ...$

Further, the use of the free surface condition leads to

$A = \dfrac{ag}{\omega \cosh kh}$ and $\omega^2 = gh \tanh kh$.

Further, $k_n = \dfrac{n\pi}{L}$ will yield various wave modes associated with the wave motion.

Two dimensional standing waves in a channel (small amplitude gravity waves)

80. Consider the propagation of standing wave of the form $\eta(x, z, t) = a\cos k_x x \cos k_z z \cos \omega t,$ in a channel of uniform depth h, length a and width b. Find the velocity potential associated with the wave motion.

Ans: Recapitulating, in this case, the velocity potential satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \qquad \text{(A)}$$

The surface boundary conditions are given by

$$\phi_t + g\eta = 0 \ \text{ on } \ y = 0, \qquad \text{(B)}$$

$$\phi_y + \eta_t = 0 \ \text{ on } \ y = 0 \qquad \text{(C)}$$

Further, the bottom boundary condition is given by

$$\phi_y = 0 \ \text{ on } \ y = -h. \qquad \text{(D)}$$

Assuming the channel walls are at $x = (0, a), \ z = (0, b)$, the wall boundary conditions are given by

$$\left.\begin{array}{l} \phi_x = 0 \ \text{ at } x = (0, a) \\ \phi_z = 0 \ \text{ at } x = (0, b) \end{array}\right\} \qquad \text{(E)}$$

Proceeding in a similar manner as in Example 1, it is derived that

$$\phi(x, y, t) = A\cosh k(h + y)\cos k_x x \cos k_z z \sin \omega t \qquad \text{(F)}$$

The wall boundary conditions yield

$$k_x = \frac{n\pi}{a}, \ \ k_z = \frac{m\pi}{b}, \ \ m, n = 1, 2, ... \qquad \text{(G)}$$

Further, substituting for ϕ from (F) in (A), it is derived that

$$k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = k_{mn} \quad \text{(say).}$$

Substituting for ϕ in the surface boundary condition, it is derived that

$$\omega^2 = gh\tanh kh$$

$$\text{and}\,\phi = \frac{ag}{\omega}\frac{\cosh k(h+y)}{\cosh kh}\cos k_x x\cos k_z z\sin\omega t$$

$$= \frac{ag}{\omega}\frac{\cosh k(h+y)}{\cosh kh}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi z}{b}\right)\sin\omega t$$

So for each $m, n,$ we have a velocity potential. Hence the superposition of all velocity potentials is also a solution which is given by

$$\phi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}g}{\omega}\frac{\cosh k_{mn}(h+y)}{\cosh k_{mn}h}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi z}{b}\right)\sin\omega t$$

Phase and group velocities for gravity waves

81. The dispersion for plane gravity wave in finite water depth is given by $\omega^2 = gk\tanh kh$
Find the relation group velocity in terms of phase velocity.
Ans: The phase velocity c c and the group velocity c_g are given by

$$c = \frac{\omega}{k} = \frac{\lambda}{T} \text{ and } c_g = \frac{d\omega}{dk},$$

Given $\omega^2 = gk\tanh kh$

$$\text{or}\, 2\omega\frac{d\omega}{dk} = g\tanh kh + gkh\sec h^2 kh$$

$$\Rightarrow c_g = \frac{d\omega}{dk} = \frac{1}{2}\frac{g}{\omega}\tanh kh\left\{1+\frac{2kh}{\sinh 2kh}\right\} = \frac{1}{2}\frac{gk\tanh kh}{k\omega}\left\{1+\frac{2kh}{\sinh 2kh}\right\}$$

$$= \frac{1}{2}\frac{\omega}{k}\left\{1+\frac{2kh}{\sinh 2kh}\right\} = \frac{c}{2}\left\{1+\frac{2kh}{\sinh 2kh}\right\}.$$

In case of deep water $kh >> 1$, Therefore $c_g = c/2$. On the other hand, in case of shallow water waves, $kh << 1$ which yield $c_g = c$.

Phase and group velocities for capillary gravity waves

82. The dispersion relation $\omega^2 = gk(1+Mk^2)\tanh kh$ associated with the plane capillary gravity waves is given by $K = k(1+Mk^2)\tanh kh,$ where $K = \omega^2/g$, $M = T/\rho g$, T being surface tension force. Find the relation connecting phase and group velocities c and c_g respectively.

Ans: From dispersion relation

$$\omega^2 = gk(1+Mk^2)\tanh kh$$

$$\text{or } 2\omega\frac{d\omega}{dk} = g\left\{(1+Mk^2)\tanh kh + kh(1+Mk^2)\sec h^2 kh\right\}$$

$$\text{or } c_g = \frac{g\omega}{2\omega^2}\left\{(1+3Mk^2)\tanh kh + kh(1+Mk^2)\sec h^2 kh\right\}$$

$$= \frac{g\omega}{2gk(1+Mk^2)\tanh kh}\left\{(1+3Mk^2)\tanh kh + kh(1+Mk^2)\sec h^2 kh\right\}$$

$$= \frac{\omega}{2\mathrm{k}}\left\{\frac{1+3Mk^2}{1+Mk^2} + \frac{2kh}{\sinh 2kh}\right\} = \frac{c}{2}\left\{\frac{1+3Mk^2}{1+Mk^2} + \frac{2kh}{\sinh 2kh}\right\}$$

Therefore, in case of deep water, $kh >> 1$ which yields $c_g = \frac{c}{2}\left\{\left(1+3Mk^2\right)/\left(1+Mk^2\right)\right\}$.

On the other hand, in case of shallow water, $c_g = \frac{c}{2}\left\{\left(1+2Mk^2\right)/\left(1+Mk^2\right)\right\}$.

Capillary gravity wave

83. In case of capillary gravity wave, find the length of the smallest possible wave in terms of surface tension parameter T. Thus, find λ for $g = 9.8\text{m/s},\ T = 0.074\text{N/m}$ and $\rho = 1000\text{kg/m}^3$.

Ans: From dispersion relation, in case of deep water $K = k(1+Mk^2) \Rightarrow \omega = gk\tanh kh$

Therefore, $c^2 = \omega^2/k^2 = g(1/k + Mk)$

$$\Rightarrow 2c\frac{dc}{dk} = g(-1/k^2 + M)$$

Therefore, $\frac{dc}{dk} = 0$

$$\Rightarrow M = 1/k^2$$

$$\Rightarrow k = 1/\sqrt{M} = \sqrt{\rho g/T}$$

Therefore, $k = \sqrt{\rho g/T}$

$$\text{and } c_m^2 = \frac{g}{\sqrt{\rho g/T}} + \frac{T}{\rho}\sqrt{\rho g/T} = \left(\frac{4Tg}{\rho}\right)^{1/2}$$

$$\Rightarrow c_m = \left(\frac{4Tg}{\rho}\right)^{1/4}$$

$$\Rightarrow \lambda_m = 2\pi/k = 2\pi(T/\rho g)^{1/2}$$

Next, for $g = 9.8\text{m/s}^2,\ \rho = 1000\text{Kg/m}^3,\ T = 0.074\text{N/m},\ c_m = 23\text{cm/s},\ \lambda_m = 1.7\text{cm}$.

This is the minimum wave length of gravity wave possible in water.

Wave oscillation in a closed basin and resonant period

84. A section of a closed basin has a depth of 8m and a horizontal length of 1000m when resonance occurs in this section at the fundamental period, the height of the standing wave is 0.2m. Determine the resonant period and wave length of the wave.

Ans: Given $h = 8\text{m},\ L = 1000\text{m},\ H = 0.2\text{m}$.

Assuming shallow water wave, the fundamental period of oscillation is given by

$$T = \frac{2L}{\sqrt{gh}} = \frac{2000}{\sqrt{9.81\times 8}} = 225.76\text{s}$$

Now, $c = \frac{\lambda}{T} \Rightarrow \lambda = cT = \sqrt{9.8\times 8\times 225.76}\text{m} = 2000\text{m}$.

Hence, $h/\lambda = \frac{8}{2000} < \frac{1}{20}$. Thus, the assumption of shallow water is justified.

Fundamental period in a one dimensional lake

85. Derive the fundamental period of oscillation for a lake of length 15miles having average depth 22ft .

Ans: Given $L = 15\text{miles}, h = 22\text{ft}, g = 32\text{ft}/\sec^2$. Further, it may be noted that $1\text{mile} = 5280\text{ft}$.

Therefore, $T = \frac{2L}{\sqrt{gh}} = \frac{2\times 5280\times 15}{\sqrt{32\times 22}} = 5968\sec = 99.5\text{min}$

Alternately, $T = \frac{2\times 15\times 1609}{\sqrt{9.8\times 22\times 0.3048}}\text{s} = 99.5\text{min}$ (using the formula 1 mile = 1.609 km).

Long wave equation in a channel of variable cross-section

86. Derive the equation of long wave in channel of variable cross-section $A(x,t)$ which is infinitely extended along the x-axis are small compared to the wavelength.

Ans: Let (u, v, w) be the components of the velocity $\vec{q}$ with u being large compared to v and w. Thus the x-component of Euler's equation of motion yields

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1)$$

and the z-component yields

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g \qquad (2)$$

where the quadratic terms are ignored under the assumption of small amplitude wave theory. Eq. (2) on integration yields

$$p = -\rho g z + f(x,t)$$

Assuming at the free surface $z=\zeta(x,t)$, the hydrodynamic pressure is the same as the constant atmospheric pressure P_0, Eq. (2) yields

$$p = p_0 + \rho g(\zeta - z) \qquad (3)$$

Next, using the value of p from Eq. (3) in Eq. (1), equation of motion for long wave is obtained as

$$\frac{\partial u}{\partial t} = -g\frac{\partial \zeta}{\partial x}. \qquad (3a)$$

Next, to derive the equation of continuity in case of long wave, consider a volume of liquid bounded by two planes cross-section of the channel at a distance dx apart. Thus in unit time, the volume of liquid through one plane of the channel is $(Au)_x$. Further the volume of liquid flowing through the other plane of the channel is $(Au)_{x+dx}$. Thus, the net change in volume of liquid flowing between the two planes per unit time is

$$(Au)_{x+dx} - (Au)_x = \frac{\partial (Au)}{\partial x}dx. \qquad (4)$$

Further, the rate of change in volume of liquid per unit time between the two planes is

$$\frac{\partial A}{\partial t}dx. \qquad (5)$$

From the law of conservation of mass, rate of change of mass within the two planes = net change in mass flowing between the two plane per unit time.Assuming the fluid is incompressible, density is taken as constant. Thus, Eq. (4) and Eq. (5) yield

$$\frac{\partial A}{\partial t} + \frac{\partial (Au)}{\partial x} = 0. \qquad (6)$$

Eq. (6) is the equation of continuity for long wave. Thus, Eqs. (3a) and (6) yield the linearized long wave equations under shallow water approximation in an infinitely channel of variable cross section.

Long wave equation in a channel of finite width b

87. Generalise the equation of continuity discussed in Ex:1 for a channel of width b

Ans: Let A_0 be the cross-sectional area of the fluid in the channel in equilibrium position. Assuming $y=\zeta(x,t)$ as the surface elevation, the change in the cross-sectional area due to wave action is

$$A' = b\zeta$$

Thus, equation (6) in Ex. 23 becomes

$$\frac{\partial (b\zeta)}{\partial t} + \frac{\partial (Au)}{\partial x} = 0$$

$$\text{or} \quad b\frac{\partial \zeta}{\partial t} + \frac{\partial (Au)}{\partial x} = 0 \qquad (7)$$

Differentiating Eq. (7) w.r.t. time t, we get

$$b\frac{\partial^2\zeta}{\partial t^2}+\frac{\partial}{\partial t}\left(\frac{\partial(Au)}{\partial x}\right)=0$$

or $$b\frac{\partial^2\zeta}{\partial t^2}+\frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial t}\right)=0$$

or $$b\frac{\partial^2\zeta}{\partial t^2}+\frac{\partial}{\partial x}\left\{A\left(-g\frac{\partial u}{\partial x}\right)\right\}=0$$

or $$b\frac{\partial^2\zeta}{\partial t^2}-g\frac{\partial}{\partial x}\left(A\frac{\partial\zeta}{\partial x}\right)=0.$$

If the channel is of uniform cross-sectional area $A=A_0$, then

$$\frac{\partial^2\zeta}{\partial t^2}-\frac{gA_0}{b}\frac{\partial^2\zeta}{\partial x^2}=0$$

Thus, the speed of propagation in channel of uniform cross section A_0 is $c=\sqrt{gA_0/b}$.

Long wave propagation in a channel

88. Derive the long wave equation is a channel of infinite length in water of finite depth, assuming wave amplitude is very small.

Ans: It is assumed that the z-component of velocity is very small compared to the x and y components of velocities.

Thus, in case of 2-D wave equation, here, the long wave equation of motion yields

$$\frac{\partial u}{\partial t}+g\frac{\partial\zeta}{\partial x}=0 \qquad \text{(A)}$$

and $$\frac{\partial v}{\partial t}+g\frac{\partial\zeta}{\partial y}=0 \qquad \text{(B)}$$

Further, proceeding in a similar manner as in case of channel of variable cross-section, the Eq.of continuity for linear long wave yield

$$\frac{\partial h}{\partial t}+\frac{\partial(hu)}{\partial x}+\frac{\partial(hv)}{\partial y}=0 \qquad \text{(C)}$$

Assuming total water depth $h=h_0+\zeta(x,y,t)$ where h_0 is the water depth from mean surface till bottom, Eq. (C) yields

$$\frac{\partial\zeta}{\partial t}+\frac{\partial(h_0u)}{\partial x}+\frac{\partial(h_0v)}{\partial z}=0 \qquad \text{(D)}$$

which in case of constant depth h_0 becomes

$$\frac{\partial\zeta}{\partial t}+h_0\frac{\partial u}{\partial x}+h_0\frac{\partial v}{\partial z}=0. \qquad \text{(E)}$$

Eliminating u and v from (A), (B) and (C), the long wave equation in water of uniform depth is obtained as

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}=\frac{1}{C^2}\frac{\partial^2\zeta}{\partial t^2}$$

where $C=\sqrt{gh_0}$.

Long wave reflection due to change in depth near a wall

89. A plane wave is reflected due to abrupt change in water depth from depth h_1 to depth h_2 which is at a finite distance l from a sea wall.

Thus, near the wall $\eta_x=0$

Therefore, $\eta(x,t)=\begin{cases}\left(e^{ikx}+\mathrm{Re}^{-ikx}\right)e-^{i\omega t}, & x<0\\ T\cos k(x-l)e^{-i\omega t}, & 0<x<l\end{cases}$

Continuity of pressure and velocity at $x=0$, yield

$1+R=T\cos kL.$

Further, $h_1ik(1-R)=h_2T\sin kL$

which yield

$$T=\frac{2\sqrt{h_1}}{\sqrt{h_1}\cos k_2L-i\sqrt{h_2}\sin k_2L},\ R=\frac{\cot k_2L+i\sqrt{h_2/h_1}}{\cot k_2L-i\sqrt{h_2/h_1}}$$

Long wave reflection by a finite dock near a wall

90. Wave reflection by a finite dock near a wall (Shallow water approximation)

A rigid dock of width a is located at a distance L from a vertical wall in uniform water depth. Assuming the at the sea wall is at $x=a+L$ and dock is located at $-a<x<a$. A shallow water plane wave is incident from $x=-\infty$

Assuming $\phi=\begin{cases}\dfrac{iga}{\omega}\left(e^{ikx}+\mathrm{Re}^{-ikx}\right)e^{-i\omega t}, & x<-a\\ \dfrac{iga}{\omega}T\cos k(x+a+L)e^{-i\omega t}, & a<x<L\end{cases}$

where R is the reflection coefficient under shallow water approximation then find R

Ans: Below the plate under shallow water approximation

$$\phi=\frac{iga}{\omega}\left(Ax+B\right)e^{-i\omega t},\qquad -a<x<a$$

Now, continuity of velocity and pressure at $x=\pm a$ yield the boundary condition

$\phi_{1x}=\phi_{2x},\ \phi_1=\phi_2$ at $x=\pm a$

Thus, $e^{-ika}+\mathrm{Re}^{ika}=-Aa+b,\ ik\left(e^{-ika}-\mathrm{Re}^{ika}\right)=A$

Further, $Aa+B=T\cos k(a+L)$

Further, $A=-kT\cos k(a+L),\ Aa+B=T\cos k(a+L),\ -Aa+B=e^{-ika}+\mathrm{Re}^{ika}$

which yields $B=\frac{1}{2}\left\{e^{-ika}+\mathrm{Re}^{ika}+T\cos k(a+L)\right\}$.

The set of equations can be solved to find R.

Long wave scattering by a finite dock

91. A two-dimensional rigid dock of width 2a which is moored in water of uniform depth. The sides of the dock at $x=\pm a$. A shallow water plane wave

$\eta = ae^{i(kx-\omega t)}$, $\quad k=\omega\big/\sqrt{gh}$ is incident from $x=-\infty$

Assuming

$$\phi=\begin{cases}\dfrac{iga}{\omega}\left(e^{ikx}+\mathrm{Re}^{-ikx}\right)e^{-i\omega t}, & x<-a\\ \dfrac{iga}{\omega}Te^{i(kx-\omega t)}, & x>a\end{cases}$$

where R and T are the reflection and transmission coefficients. Find R and T.

Now below the plate, under shallow water approximation

$$\phi=-\frac{iga}{\omega}(ax+b)e^{-i\omega t},\quad -a<x<a$$

Using the continuity of mass and pressure at $x=\pm a$, we obtain $\phi_{1x}=\phi_{2x}$, $\quad \phi_1=\phi_2$ at $x=\pm a$

Thus,
$$e^{-ikx}+\mathrm{Re}^{ikx}=-\alpha x+b$$
$$ik\left(e^{-ika}-\mathrm{Re}^{ika}\right)=\alpha.$$

Further,
$$T\,\mathrm{e}^{ikx}=\alpha x+b$$
$$ikTe^{ika}=\alpha.$$

Therefore, $R=\dfrac{ikae^{-2ika}}{1-ika}$, $\quad T=\dfrac{e^{-2ika}}{1-ika}$, $\alpha=\dfrac{ike^{-ika}}{1-ika}$, $\quad \beta=e^{-ika}$.

Particle kinematics of a plane gravity wave

92. For a plane wave $\eta=a\cos(kx-\omega t)$, find the components of particle speed and thus find the speed of the particle in case of deep water.

Ans: In finite water depth, $\eta=a\cos(kx-\omega t)$.

$$\text{Thus, }\phi=\frac{ag}{\omega}\frac{\cosh k(h+y)}{\cosh kh}\sin(kx-\omega t)\quad =\frac{ag\omega}{\omega^2}\frac{\cosh k(h+y)}{\cosh kh}\sin(kx-\omega t)$$
$$=\frac{ag\omega}{gk\tanh kh}\frac{\cosh k(h+y)}{\cosh kh}\sin(kx-\omega t)=\frac{a\omega}{k}\frac{\cosh k(h+y)}{\sinh kh}\sin(kx-\omega t).$$

$$\phi_x=\frac{a\omega k}{k}\frac{\cosh k(h+y)}{\sinh kh}\cos(kx-\omega t)=\frac{\mathrm{H}\pi}{T}\frac{\cosh k(h+y)}{\sinh kh}\cos(kx-\omega t).$$

$$\phi_y=\frac{a\omega k}{k}\frac{\sinh k(h+y)}{\sinh kh}\sin(kx-\omega t)=\frac{\mathrm{H}\pi}{T}\frac{\sinh k(h+y)}{\sinh kh}\sin(kx-\omega t),$$

In case of deep water, $kh >> 1$. Thus

$$\phi_x = \frac{H\pi}{T} e^{ky} \cos(kx - \omega t),\ \phi_y = \frac{H\pi}{T} e^{ky} \sin(kx - \omega t) \Rightarrow q = \sqrt{\phi_x^2 + \phi_y^2} = \frac{H\pi}{T} e^{ky}.$$

Thus, near the free surface at $y = 0$, the particle speed is given by $q = \pi H/T$. Thus, each component of velocity has three parts, namely (i) the surface deep water particle speed, (ii) particle velocity variation over the vertical water column at a given location and (iii) passing term dependent position in wave and time.

93. For the wave profile $\eta = H \cos kx \sin \omega t$

In a channel of depth h, determine the maximum water particle velocity and below the nodal point under small amplitude shallow water approximation.

Ans: $\eta = H \cos kx \sin \omega t$

$$\phi = \frac{gH}{2\omega} \frac{\cos k(h+z) \cos kx \sin \omega t}{\cosh kh}$$

$$u = \frac{\partial \phi}{\partial x} = \frac{gHk}{2\omega} \frac{\cos k(h+z)}{\cosh kh} \sin kx \sin \omega t$$

$$= \frac{H}{2} \frac{2\pi}{T} \left| \frac{\cos k(h+z)}{\sinh kh} \right| \sin kx \sin \omega t = \frac{\pi H}{T} \frac{1}{kh} \sin kh \sin \omega t.$$

For peak velocity under shallow water depth the nodal point, $\sin kx \sin \omega t = 1$,

$$U_{\max} = \frac{\pi H}{T} \frac{\lambda}{2\pi h} = \frac{H\lambda}{2hT} = \frac{H}{2h} c = \frac{H}{2h} \sqrt{gh} = \frac{H}{2} \sqrt{\frac{g}{h}}.$$

Long wave resonance in a bay

94. Under shallow water approximation, find the condition of resonance in a bay.

Ans: Under shallow water approximation, one dimensional wave equation is given by

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \eta}{\partial t^2}$$

Near a bay head, standing waves are formed. Thus $\eta = \cos kx \cos \omega t$

$$\frac{\eta(0)}{\eta(l)} = \left| \frac{1}{\cos kl} \right|$$

Near bay mouth nodes are formed and near bay head antinodes are formed. Assuming bay mouth is at $x = 0$ and bay head is at $x = l$, we have

During resonance $\frac{\eta(0)}{\eta(l)} \to \infty \Rightarrow \cos kl = 0 = \cos(2n+1)\frac{\pi}{2} \Rightarrow kl = (2n+1)\frac{\pi}{2}$

$$\Rightarrow \frac{2\pi}{\lambda} l = (2n+1)\frac{\pi}{2} \Rightarrow l = (2n+1)\frac{\lambda}{4} \Rightarrow \lambda = \frac{4l}{(2n+1)}.$$

Now, for long wave $c = \lambda/T = \sqrt{gh} \Rightarrow T = \frac{\lambda}{\sqrt{gh}} = \frac{4l}{(2n+1)\sqrt{gh}},\ 1, 2, \ldots$

Hence, when $n = 0$, $T = \frac{4l}{\sqrt{gh}}$ is the fundamental period for bay oscillation.

Long wave oscillation in a closed basin

95. Derive the relation for period of oscillation in case of long waves propagating in a closed basin of length l.

Ans. Under the assumption of linearized longwave theory, the free surface elevation η satisfies

$$\frac{\partial^2\eta}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2}, \qquad c=\sqrt{gh}$$

On the channel side walls $\eta_x=0$ at $x=0,\ l$

Assuming $\eta(x,t)=a\cos kx\cos\omega t$ is a stationary wave solution of Eq. (A), the wall boundary condition yield

$$\sin kl=0=\sin n\pi,\ n=1,2,...\Rightarrow kl=n\pi$$

$$\Rightarrow l=\frac{n\pi\lambda}{2\pi}=\frac{n\lambda}{2}\ \Rightarrow\ \lambda=\frac{2l}{n},\ n=1,2,..$$

Therefore, the phase velocity

$$c=\frac{\lambda}{T}=\sqrt{gh}\Rightarrow T=\frac{\lambda}{\sqrt{gh}}=\frac{2l}{n\sqrt{gh}},$$

Hence, $T_n=\dfrac{2l}{n\sqrt{gh}}$ are the period of oscillation.

96. A section of a closed basin has a depth of 20m and a horizontal length of 2000m. When resonance occures in this section at the fundamental period, the height of the standing wave is 0.5m. Determine the resonant period, the maximum water particle velocity under the nodal point.

Ans: The fundamental period of oscillation also called the resonant period is given by

$$T_n=\frac{2l}{\sqrt{gh}}=\frac{2\times 200}{\sqrt{9.8\times 20}}=\frac{4000}{\sqrt{196}}=285.71\text{sec}.$$

Further, maximumwater particle velocity

$$u_{\max}=\frac{H}{2}\sqrt{\frac{g}{h}}=\frac{0.5}{2}\sqrt{\frac{9.8}{20}}=0.25\sqrt{0.47}\approx 0.55m/s.$$

Wave oscillation in a rectangular tank under shallow water wave approximation

97. Determine the period of free oscillation in a rectangular lake of uniform water depth, length and width h, a, and b respectively under shallow water approximation.

Ans: The two-dimensional linearizedlong wave equation is given by

$$\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2}, \quad c=\sqrt{gh} \qquad (1)$$

Assuming motion is simple harmonic in time with angular frequency ω, $\bar{\eta}(x,y,t)$ is written as $\bar{\eta}(x,y,t)=\eta(x,y)e^{-i\omega t}$. Thus, Eq.(1) yields

$$\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}+k^2\eta=0 \qquad (2)$$

where $k = \omega/c$. Assuming that the walls are located at $x = 0, a$ along length and $y = 0, b$ along width, the wall boundary conditions yield

$$\frac{\partial \eta}{\partial x} = 0 \quad \text{at} \quad x = 0,\ a \qquad (3)$$

and $$\frac{\partial \eta}{\partial y} = 0 \quad \text{at} \quad x = 0,\ \text{b}. \qquad (4)$$

The solution of (2) satisfying boundary conditions given in (3) and (4) is obtained as

$$\eta(x, y) = \sum_{m=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad (5)$$

Now substituting for η from (5) in (2), yields $k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$

Thus, the time period is given by

$$T = \frac{\lambda}{c} = \frac{\lambda}{\sqrt{gh}} = \frac{2\pi}{k\sqrt{gh}} = \frac{2\pi}{\sqrt{gh}\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}} = \frac{2}{\sqrt{\text{gh}}}\frac{1}{\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}},$$

where m and n are referred as the modes of oscillation along the length and width of the channel.

Wave oscillation in a circular lake under shallow water approximation

98. Under shallow water approximation, find the relation for the wave period during resonance in a circular lake of radius *a* and depth *h*.

Ans: Consider a circular lake of radius *a* and depth *h*. The two dimensional wave equation is given by

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \eta}{\partial t^2}, \quad c = \sqrt{gh} \qquad \text{(A)}$$

In cylindrical polar co-ordinate, Eq. (A), is rewritten as

$$\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2} + k^2\eta = 0\ , \qquad \text{(B)}$$

under the assumption $\bar{\eta}(r, \theta, t) = \eta(r, \theta)e^{-i\omega t}$.

Assuming that the oscillating motion is periodic in θ, $\eta(r, \theta)$ is rewritten as

$\eta(r, \theta) = f(r)e^{is\theta}$.

Thus, Eq. (B) yield

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \left(k^2 - \frac{s^2}{r^2}\right) f = 0 \,\text{(C)}$$

Writing $K = kr$, Eq.(C) can be rewritten as

$$\frac{\partial^2 f}{\partial k^2}+\frac{1}{k}\frac{\partial f}{\partial k}+\left(1-\frac{s^2}{k^2}\right)f=0 \qquad \text{(D)}$$

Assuming that the wave amplitude is bounded near $r=0$, Eq. (D) yield

$$\eta(r,\theta)=A_s J_s(kr)\begin{Bmatrix}\cos s\theta\\ \sin s\theta\end{Bmatrix}$$

On the boundary wall of the cylinder

$\frac{\partial\eta}{\partial r}=0$ at $r=a$, which gives

$J_s^{'}(ka)=0 \quad$ for $s=0,1,21\cdots$

In particular, for $s=0$

$J_0^{'}(ka)=0 \Rightarrow J_0^{'}(ka)=0,$ which yield

$$\frac{2\pi a}{\lambda}=ka=1.2197, 2.2330, 3.2383$$

$$\Rightarrow \frac{2\pi a}{\lambda}=0$$

We know that $\lambda=cT \Rightarrow T=\lambda/c=\lambda/\sqrt{gh}$ (E)

Hence, from (D) and (E),

$$T_1=\frac{2a}{1.2197\sqrt{gh}},\ T_2=\frac{2a}{2.233\sqrt{gh}},\ T_3=\frac{2a}{3.2383\sqrt{gh}}$$

Free surface flow inside a rotating circular cylinder

99. Determine the flow pattern inside a cylinder which rotates about its vertical axis with constant angular velocity Ω and the surface being open to the atmosphere. (Assumethat the atmospheric pressure is constant and the flow is in the gravitational field).

Ans: Assume that the axis of the cylinder is along the z-axis. Thus, the components of velocity are $u=-y\Omega,\quad v=x\Omega,\quad w=0.$ Thus, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

which ensures the flow of an incompressible fluid. Hence, Euler equation yields

$$x\Omega^2=\frac{1}{\rho}\frac{\partial p}{\partial x},\quad y\Omega^2=\frac{1}{\rho}\frac{\partial p}{\partial y},\quad \frac{1}{\rho}\frac{\partial p}{\partial z}+g=0$$

which on integration yield

$\frac{p}{\rho}=\frac{1}{2}\Omega^2\left(x^2+y^2\right)-gz+c,$ where c is an arbitrary constant.

Since $p=p_{atm}=$ constant, at $z=0$ at every point on the free surface. It is also zero at $(x,y)=(0,0)$. Hence $\frac{p}{\rho}=c.$

$\Rightarrow z = \dfrac{1}{2g}\left(x^2 + y^2\right)$ is the surface of fluid.

Further, the streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \Rightarrow \frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{0}$$

$\Rightarrow x^2 + y^2 = \text{constant}, z = \text{constant}$

gives the streamlines for the flow.Further, $\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 2\Omega$ yields the vorticity vector which is along the z-axis.

Circular Couette flow (Steady viscous flow between two concentric rotating cylinders)

100.Describe the steady viscous flow between two concentric cylinders which are rotating at different angular velocity.

Consider the steady viscous flow between two concentric cylinders. Let R_1 and Ω_1 be the radius and angular velocity of the inner cylinder and R_2 and Ω_2 be the radius and angular velocity of the outer cylinder.Thus, the equations of motion in the radial and tangential dimensions are given by

$$\frac{u_0^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{(A)}$$

$$\mu \frac{d}{dr}\left[\frac{1}{r}\frac{d\left(ru_\theta\right)}{dr}\right] = 0. \qquad \text{(B)}$$

Integrating (B) yield

$$u_\theta = Ar + \frac{B}{r}$$

Using the boundary conditions

$u_\theta = \Omega_1 R_1$ at $r = R_1$

$u_\theta = \Omega_2 R_2$ at $r = R_2$

which gives $A = \dfrac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}$, $B = \dfrac{\left(\Omega_1 - \Omega_2\right) R_1^2 R_2^2}{R_2^2 - R_1^2}$.

Thus, u_θ is obtained as $u_\theta = \dfrac{1}{1-\left(R_1/R_2\right)^2}\left\{\left[\Omega_2 - \Omega_1\left(R_1/R_2\right)^2\right]r + \dfrac{R_1^2}{r}\left(\Omega_1 - \Omega_2\right)\right\}$.

This flow is referred as the circular Couette flow.

Note. For an incompressible fluid, the viscous stress at a point is given by

$$\sigma_{ij} = u\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

which shows that σ depends only in the deformation rate of the fluid element at a point and not on the rotation rate $\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$. On the other hand, the net viscous force per unit volume at a point is given by $F_i = \frac{\partial \sigma_{ij}}{\partial x_j} = -u(\nabla \times w)$.

Prove that the fluid elements in a solid body rotation do not deform. Thus, prove that surface of constant pressure are paraboloids of revolution.

Ans: Consider the velocity vector associated with a solid body rotation

$u_\theta = \omega r/2$ and $u_r = 0$.

The viscous stress $\sigma_{r\theta}$ is given by

$$\sigma_{r\theta} = \mu\left\{\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right\} = 0$$

Thus, the fluid elements in a solid body rotation do not deform.

Since the viscous stress vanishes, Euler's Equation of motion is applied for the flow problem. Euler's Equation of motion in cylindrical polar co-ordinate is given by

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{(A)}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad \text{(B)}$$

$$\frac{\partial u_z}{\partial t} + (u\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \rho g \qquad \text{(C)}$$

Substituting for $u_r = 0, \quad u_\theta = \frac{\omega r}{2}, \quad u_z = 0$

Eqs.(A), (B) and (C) yields

$$\rho\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r}, \qquad \frac{\partial p}{\partial \theta} = 0, \ \frac{\partial p}{\partial z} + \rho g = 0$$

Now $dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz = \rho\frac{u_\theta^2}{r}dr - \rho g dz = \frac{\rho r\omega^2}{4}dr - \rho g dz$

$$\Rightarrow \int_{p_1}^{p_2} dp = \frac{\rho w^2}{4}\int_{r_1}^{r_2} r dr - \rho g\int_{z_1}^{z_2} dz$$

$$\Rightarrow p_2 - p_1 = \frac{\rho\omega^2}{8}\left\{\left(\frac{r_2^2 - r_1^2}{2}\right) - \rho g(z_2 - z_1)\right\} \qquad \text{(D)}$$

Thus, for $p_2 = p_1$,

$$\Rightarrow z_2 - z_1 = \frac{\omega^2}{8g}\left(r_2^2 - r_1^2\right)$$ which is a paraboloid of revolution.

Therefore surface of constant pressure are paraboloid of revolution in case of fluid elements in a solid-body rotation. It may be noted that the flow is steady in this case.

Note: Eq.(D) can be rewritten as

$$p_2 - p_1 = \frac{\rho}{2}\left(u_{\theta 2}^2 - u_{\theta 1}^2\right) - \rho g(z_2 - z_1)$$

$\Rightarrow p_2 - \frac{\rho}{2}u_{\theta 2}^2 + \rho g z_2 = p_1 - \frac{\rho}{2}u_{\theta 1}^2 + \rho g z_1$ which suggests that in case of solid body rotation containing a viscous fluid in steady that $p/\rho - u_\theta^2 + gz$ is not a constant for points of different streamlines.

Example of viscous fluid with irrotational motion

101.Show that the flow does not have any singularity in the entire field and is irrotational everywhere. Viscous stresses are present and no net viscous force at any point in the steady state. Here, the flow field is viscous but irrotational.

Ans: Consider the flow field is given by

$u_\theta = \frac{\Omega R^2}{r}, \; u_r = 0, \; u_z = 0, \; r \geq R,$ which is the velocity distribution of an irrotational vortex.

Here $\Omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\sigma_{r\theta} = 0$

$$\Omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 0$$

$$\Omega_z = \frac{1}{r}\frac{\partial\left(r u_\theta\right)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{r}\left(r\frac{\Omega R^2}{r}\right) - \frac{1}{r}\frac{\partial (0)}{\partial \theta} = 0$$

Therefore $\vec{\Omega} = (\Omega_{x,}\; \Omega_{y,}\; \Omega_z) = 0$

This implies that the flow field is irrotational.

Next, by definition of viscous stress

$$\tau_{r\theta} = \mu\left\{\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right\} = \frac{2\mu\Omega R^2}{r^2} \neq 0$$

Thus, shear stress exists. However, net viscous force per unit volume at a point is given by

$$F_i = -\mu\left(\nabla \times \vec{\Omega}\right) = 0.$$

Hence, the net force per unit volume at a point is zero. This is the flow generated due the presence of a rotating cylinder in an infinite volume of viscous fluid which rotates with constant angular velocity $\Omega/2$. The above velocity field is also called the steady solution of the N-S equation satisfying the no-slip boundary condition $u_\theta = \Omega R$ at $r = R,$ with R being the radius of the cylinder and Ω is its angular velocity. This flow is not singular.

Steady viscous flow outside a circular cylinder rotating in an infinite body of fluid

102.Find the velocity distribution of a steady viscous flow outside a long circular cylinder of radius R which is rotating with angular velocity Ω in an infinitely body of fluid.

Ans: The result directly follows from the general solution discussed in the previous example with $\Omega_2 = 0,\ R_2 = \infty,\ \Omega_1 = \Omega$ and $R_1 = R$. This gives $u_\theta = \Omega R^2/r, r > R,$ Here there is no singularity in the flow and flow is irrotational in nature. This flow suggests that absence of viscous dissipation.

Ans. See the previous exercise for details

103.Consider the flow generated by rotating a solid circular cylinder of radius r in an infinite viscous fluid whose velocity field is given by $u_\theta = \left(\omega R^2\right)/2r,\ r \ge R,\ \ u_r = 0,\ u_z = 0$ where R is the radius of the cylinder and $\omega/2$ is its constant angular velocity.

Ans. See the two previous exercises.

Viscous flow within a steadily rotating cylindrical tank

104.Consider a steady rotation of a cylindrical tank containing a viscous fluid. The radius of the cylinder is R and the angular velocity of rotation is Ω. The flow would reach a steady state after the initial transients have decayed.

Ans.Thus in this case $\Omega_1 = 0$ at $R_1 = 0$

. $\Omega_2 = \Omega$ and $R_2 = R$ This gives from Ex. 1 $u_\theta = \Omega r$ which says that the tangential velocity is proportional to r .

Viscous flow generated by rotating a circular cylinder (rotational flow)

105.Discuss the flow for the vector field given by $u_r = 0,\ u_\theta = \omega_0 r,\ u_2 = 0.$

Ans: Equation of continuity yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

Thus, an incompressible fluid flow is possible. Now, the component of vorticity vector about z-axis is given by

$$w_z = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} = 2\omega_0 \ne 0.$$

Thus, the flow is rotational in nature. Equations of streamlines are

$$\frac{dr}{u_r} = \frac{rd\theta}{u_\theta} = \frac{dz}{u_z}, \text{or } \frac{dr}{0} = \frac{rdr}{\omega_0 r} = \frac{dz}{0}$$

$\Rightarrow \omega_0 rdr = 0 \Rightarrow r^2$=constant $\Rightarrow r =$ constant $\Rightarrow x^2 + y^2 =$ constant, z = constant.

Thus, the streamlines are circles.Hence, the flow velocity is proportional to the radius of the streamlines.Now the circulation around a circuit of radius r in this flow is given by

$$\Gamma = \oint \vec{q}.ds = \int_0^{2\pi} ru_\theta d\theta = 2\pi ru_\theta = 2\pi r^2 \omega_0$$

Thus, the circulation is equal to vorticitytimes area. Thus, the flow can be generated by rotating a cylindrical tank containing an incompressible fluid.

Irrotational vortex

106.Prove that in an irrotational vortex given by $u_\theta = \Gamma/2\pi r$, the viscous stress is non-zero everywhere, whilst the net viscous force on an element vanishes.(Here the flow is irrotational everywhere except at origin)

Ans: The velocity field associated with an irrotational vortex is given by

$$u_\theta = \frac{\Gamma}{2\pi r}, \quad u_r = 0, \quad u_z = 0,$$

where Γ is the circulation around a contour of radius r. Thus viscous stress for the is given by

$$\sigma_{r\theta} = \mu \left\{ \frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) \right\} = \frac{\mu\Gamma}{\pi r^2} \neq 0$$

Thus, the fluid elements undergo deformation.

The net viscous force per unit volume is given by

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_{ij}} = -\mu\left(\nabla \times \omega\right)_i = \frac{\partial^2 u}{\partial x_j \partial x_j} = 0$$

This implies that deformation of fluid element is zero everywhere in case of an irrotational vortex.

Further, from previous exercise, we have

$$\frac{\partial p}{\partial r} = \rho\frac{u_\theta^2}{r}, \quad u_\theta = \frac{\Gamma}{2\pi r}, \quad \frac{\partial p}{\partial \theta} = 0, \quad \frac{\partial p}{\partial z} + \rho g = 0$$

Therefore $dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz = \rho\frac{\partial u_\theta^2}{r}dr - \rho g dz = \frac{\rho}{r}\left(\frac{\Gamma}{2\pi r}\right)^2 dr - \rho g dz$

Integrating between any two elements yields

$$p_2 - p_1 = \frac{\rho\Gamma^2}{2\pi^2}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) - \rho g(z_2 - z_1)$$

$$\Rightarrow \frac{p_2}{\rho} + \frac{u_{\theta 2}^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{u_{\theta 1}^2}{2} + gz_1 \qquad \text{(A)}$$

Thus between any two points in the flow field, in case of irrotational vortex, the sum of pressure head and gravitational head is constant.

For $p_2 = p_1$, Eq.(A) yield

$$z_2 - z_1 = \frac{u_{\theta 1}^2}{2g} - \frac{u_{\theta 2}^2}{2g}$$

which are hyperboloid of revolution of second degree.

---THE END---